Math 53 syllabus

The following is a list of the topics to be covered in Math 53. For each topic, there is a "reading guide" to the corresponding sections of the textbook (Stewart: Calculus, Early Transcendentals, 7th edition), as well as a list of "learning goals". Although this list of goals is not 100 percent comprehensive, and ideally you should be acquiring a command of the material which cannot entirely be boiled down to a simple list of individual items, nonetheless, if you can master all of the items listed in the "learning goals" then you should be in pretty good shape.

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1 Geometric preliminaries

In multivariable calculus, one works with functions which are defined on subsets of the plane and three-dimensional space. (One can consider higher dimensions too, but we will not do that in this course.) To prepare for this, in Part 1 of the course we will first study some geometry in two and three dimensions. Specifically, we will study parametrized curves, polar coordinates, vectors, lines, planes, quadric surfaces, and vector-valued functions. The reading consists of most of Chapters 10, 12, and 13 of Stewart.

1.1 Introduction to the course, parametrized curves

Reading: Stewart sections 10.1 and 10.2. Notes: (1) The book uses the term "parametric curves", while I prefer to call these "parametrized curves", for reasons explained in lecture. (2) In section 10.1 and elsewhere, the book refers to graphing devices. Graphing devices are not required for this course; however if you have one, then playing with it may be useful.

Learning goals:

- Understand what a *parametrized* curve is, as opposed to just a curve.
- Write down a formula for a parametrized curve from a description.
- Sketch a parametrized curve from the formula.
- Calculate the slope of a parametrized curve.
- Calculate the area under a parametrized curve without vertical tangents, with the correct sign.
- Calculate the length of a parametrized curve.

- Calculate the area of the surface obtained by rotating a parametrized curve around the x or y axis.

1.2 Polar coordinates

Reading: Stewart sections 10.3 and 10.4. Note: It is not required to read sections 10.5 and 10.6, on conic sections. We will later study quadric surfaces, which are a three-dimensional analogue of conic sections, so it might be useful to review conic sections a little in section 10.5. However sections 10.5 and 10.6 give a lot of details about conic sections which we will not use later.

Learning goals:

- Convert between polar coordinates and Cartesian coordinates.
- Sketch curves of the form $r = f(\theta)$, watching out for the range of θ and situations where $r < 0$.
- Calculate slope and length of curves in polar coordinates.
- Calculate areas in polar coordinates, being careful to get the correct region.

1.3 Three-dimensional space, vectors, dot product, cross product

Reading: Stewart sections 12.1, 12.2, 12.3, and 12.4. Notes: As explained in lecture, the "proof" of the distance formula in 12.1 is not really a mathematically rigorous proof, because it makes a number of hidden assumptions. Rather, the distance formula can be regarded as one of the axioms defining three-dimensional Euclidean geometry, and the "proof" in the book can be regarded as motivation for assuming this axiom. The "proof" of the geometric interpretation of dot product in section 12.3 is also problematic, because it assumes the Law of Cosines, and we don't yet know that this law is true (or even what angles mean) in three dimensions. We will give a better proof in lecture. Finally, you can skip the part about "direction angles" and "direction cosines" in section 12.3.

Learning goals:

- Know the distance formula for Euclidean geometry in two and three dimensions, with the awareness that there are other, non-Euclidean geometries, e.g. spherical.

- For each of the vector operations (length, addition, multiplication by scalars, dot product, determinant, cross product), compute it and understand its geometric meaning.
- Algebraically manipulate expressions involving vector operations, avoiding undefined operations (e.g. the sum of a vector and a scalar).

1.4 Lines, planes, and quadric surfaces

Reading: Stewart sections 12.5 and 12.6. Note: It is not necessary to memorize the different quadric surfaces. But you should be able to sketch them from the equations.

Learning goals:

- Write the equation for a line in parametrized form, given a point on the line and a tangent vector to the line.
- Write the equation for the line or line segment between two points.
- Write the equation of a plane in the form $ax + by + cz + d = 0$ given a point on the plane and a normal vector, and vice versa.
- Find the equation of the plane through three points.
- Be familiar with the different kinds of quadric surfaces.
- Sketch (not rotated, but possibly translated) quadric surfaces from the equations. Highly accurate drawings are not required.

1.5 Vector-valued functions and space curves

Reading: Stewart sections 13.1 and 13.2. Note: Reading sections 13.3 and 13.4 should not be necessary. We do introduce the notions of "length", "velocity vector", and "acceleration" in lecture, and these are defined in the book only in sections 13.3 and 13.4. However these definitions are straightforward, and sections 13.3 and 13.4 mainly explain additional material, such as curvature and Kepler's laws, which we will not cover (although it is interesting to read about this if you have extra time).

Learning goals:

- Understand how to write parametrized curves in space as vector-valued functions.
- Know the definition of velocity vector and acceleration of a parametrized curve.

- Compute the tangent line to a parametrized curve at a point.
- Compute the length of a parametrized curve in space.
- Know the rules of calculus for vector-valued functions, namely three versions of the product rule and an analogue of the fundamental theorem of calculus.

2 Differentiation

Part 2 of the course introduces differentiation in multivariable calculus. Specifically, we will learn about partial derivatives and their properties and significance. At the end of Part 2 we will use partial derivatives to solve optimization problems for functions of two or three variables. The reading consists of most of Chapter 14 of Stewart.

2.1 Functions of several variables; limits and continuity

Reading: Stewart sections 14.1 and 14.2.

Note: Section 14.1 introduces the Cobb-Douglas production function, and this is revisited as an example later in Chapter 14. You are not responsible for learning about this function (but are welcome to read about it if you find it to be a useful example).

Learning goals:

- Sketch the graph of a function of two variables.
- Sketch the level sets of a function of two or three variables. Understand how some properties of a function are reflected in the appearance of its level sets.
- Be familiar with the epsilon-delta definition of a limit. (Writing proofs using epsilons and deltas will not be required for this class.)
- Know the basic limit properties and the definition of a continuous function.
- Use limit properties and continuity to compute limits when they exist.
- Prove in some cases that the limit of a function as $(x, y) \to (a, b)$ does not exist by considering the limits along different curves approaching (a, b).

2.2 Partial derivatives, tangent planes, linear approximation

Reading: Stewart sections 14.3 and 14.4.

Note: The discussion of differentials in Section 14.4 has serious issues and does not agree with standard mathematical usage. I recommend that you ignore this part to avoid confusion.

Learning goals:

- Know the definition of partial derivatives.
- Compute partial derivatives.
- Know Clairaut's theorem.
- Compute partial derivatives of functions that are defined implicitly in basic examples.
- Determine the tangent plane to the graph of a function of two variables at a point.
- Understand that a function is "differentiable" at a point when the graph is well approximated by the tangent plane.
- Use linear approximation to approximate the value of a function of two variables.

2.3 The chain rule

Reading: Stewart section 14.5.

Learning goals:

- Perform calculations using the different versions of the chain rule in multivariable calculus, and know where to evaluate the various functions that appear in the formulas.
- Calculate partial derivatives of functions defined implicitly in the general case.
- Know the statement of the Implicit Function Theorem.

2.4 Directional derivatives and the gradient vector

Reading: Stewart section 14.6.

Learning goals:

- Know the definition of directional derivatives and gradient.
- Know the geometric interpretation of the gradient, namely that it is perpendicular to the level sets and points in the direction in which the directional derivative is largest.
- Know the vector version of multivariable chain rule #1.
- Use the gradient to find the tangent line/plane to a level set of a function of two or three variables.

2.5 Maxima and minima

Reading: Stewart section 14.7.

Note: Theorem 2 is only true when (a, b) is in the interior of the domain on which f is defined. f can have a local minimum or maximum on the boundary of the domain without both partial derivatives vanishing.

Learning goals:

- Know the definitions of local and global minima and maxima, and critical points.
- Know the second derivative test for functions of two variables.
- Know the statement of the Extreme Value Theorem and understand why the hypotheses of "closed", "bounded", and "continuous" are necessary.
- Find global minima and maxima of a function on a domain by looking for critical points (and points where the partial derivatives are not both defined) and minima and maxima on the boundary.

2.6 Lagrange multipliers

Reading: Stewart section 14.8.

Note: the book doesn't explain the secret of what the Lagrange multiplier λ actually measures. I will explain this in lecture.

Learning goals:

- Understand why the method of Lagrange multipliers works.
- Solve constrained optimization problems using Lagrange multipliers.
- Know the significance of the Lagrange multiplier λ.

3 Integration

Part 3 of the course introduces integration of functions over two- and three-dimensional regions and its signifance. Unlike in single variable calculus, it is now nontrivial to work with the geometry of the regions to be integrated over. We will also see how to convert integrals to polar, cylindrical, and spherical coordinates, as well as other coordinate systems.

3.1 Basics of double integrals

Reading: Stewart sections 15.1, 15.2, and 15.3.

Note: We will not cover the Midpoint Rule from section 15.1, but you can read about this if you are interested.

Learning goals:

- Know the definition, interpretation, and basic properties of double integrals.
- Compute double integrals over rectangles using Fubini's theorem, treating the outer variable as a constant in the inner integral.
- Compute double integrals over more general regions, using the geometry of the region to write the correct limits of integration.
- Change the order of integration in double integrals.

3.2 Double integrals in polar coordinates, and surface area

Reading: Stewart sections 15.4, 15.5, and 15.6.

Note: The only applications of double integrals that we will cover are mass and center of mass. Reading about the other applications in section 15.5 (moment of inertia, etc.) is optional.

Learning goals:

- Calculate double integrals in polar coordinates.
- Understand why the magnification factor r is needed for double integrals in polar coordinates.
- Convert double integrals from Cartesian coordinates to polar coordinates and vice-versa.
- Calculate the area of a surface which is the graph of a function of two variables.

3.3 Triple integrals

Reading: Stewart section 15.7.

Note: Again, we will not cover any applications except for mass and center of mass.

Learning goals:

- Evaluate triple integrals over rectangles.
- Evaluate triple integrals over more general regions.
- Change the order of integration in triple integrals, without drawing any pictures, by logically reasoning through inequalities.
- Calculate the center of mass of a solid with varying mass density.

3.4 Triple integrals in cylindrical and spherical coordinates

Reading: Stewart sections 15.8 and 15.9.

Learning goals:

- Know the cylindrical and spherical coordinate systems.
- Convert points and regions between Cartesian, cylindrical, and spherical coordinates.
- Evaluate triple integrals in cylindrical and spherical coordinates using the correct magnification factors.

3.5 Change of variables, Jacobians

Reading: Stewart section 15.10.

Note: the book talks about "one-to-one" transformations a bit carelessly. The word "one-to-one" is a synonym for "injective"; but one also wants the transformations to be surjective. I attempted to clarify this in the second lecture segment.

Learning goals:

- Know the definition of injections, surjections, and bijections.
- Know the definition and significance of the Jacobian of a transformation.

- Know the change of variables formula for integrals in 2 and 3 variables.
- Execute a change of variables by working out the change in the geometry of the region and computing the Jacobian.
- Find a change of variables to make an integral simpler, in easy cases.

4 Vector calculus

This fourth and last part of the course brings together most of what we have done so far. We will define integrals over parametrized curves and surfaces in the plane and in space, and we will learn about four multivariable analogues of the fundamental theorem of calculus: the Fundamental Theorem of Line Integrals, Green's Theorem, Stokes' Theorem, and the Divergence Theorem.

4.1 Vector fields and line integrals

Reading: Stewart sections 16.1, 16.2.

Learning goals:

- Definition of vector fields and conservative vector fields.
- How to find a potential function for a conservative vector field by integration.
- Definition of a line integral with respect to arc length, its significance, and how to compute it.
- The definition of a line integral with respect to x or y and how to compute it.
- The definition of a line integral of a vector field, its significance, and how to compute it.

4.2 Fundamental theorem of line integrals

Reading: Stewart section 16.3.

Learning goals:

- Statement and proof of the fundamental theorem of line integrals.
- How to use the fundamental theorem of line integrals to evaluate line integrals of conservative vector fields.

- A vector field is conservative if and only if its line integral along every closed curve is zero.

4.3 Green's theorem

Reading: Stewart section 16.4.

Learning goals:

- Intuitive definition of a simply connected domain in $\mathbb{R}^2$, and statement of the Jordan curve theorem.
- Statement of Green's theorem.
- A vector field $\langle P, Q \rangle$ on a simply connected domain is conservative if and only if $Q_x = P_y$.
- Example of a nonconservative vector field $\langle P, Q \rangle$ on a non-simply connected domain such that $Q_x = P_y$.

4.4 Curl and divergence

Reading: Stewart section 16.5.

Note: you do not need to read the part after equation (12); we will explain this later when we discuss the divergence theorem.

Learning goals:

- Definition and physical interpretation of curl.
- How to decide whether a vector field on $\mathbb{R}^3$ is conservative, and when it is, how to find a potential function.
- Definition and physical interpretation of divergence.
- Curl of a gradient is zero; divergence of a curl is zero; divergence of gradient is the Laplacian.

4.5 Parametrized surfaces and surface integrals

Reading: Stewart sections 16.6 and 16.7.

Learning goals:

- Definition of a parametrized surface.
- How to go between a description of a surface and a parametrization of it.
- How to find the tangent plane to a parametrized surface at a point on the surface (where the parametrization is smooth).
- How to calculate the area of a parametrized surface.
- Definition of integration over a surface with respect to surface area.
- Definition of orientation of a surface.
- Definition of integration of a vector field over an oriented surface (flux).

4.6 Stokes' theorem

Reading: Stewart section 16.8.

Learning goals:

- Statement of Stokes' theorem.
- A differentiable vector field on $\mathbb{R}^3$ is conservative if and only if its curl is zero.
- Use Stokes' theorem to evaluate the line integral of a vector field around a closed curve by finding an oriented surface bounded by the curve and then integrating the curl over the surface.

4.7 The divergence theorem, conclusion

Reading: Stewart sections 16.9 and 16.10. Also see the end of section 16.5.

Learning goals

- Statement of the divergence theorem.
- Use the divergence theorem to evaluate the flux of a vector field through a surface without boundary.
- Use the divergence theorem to simplify the calculation of the flux of a vector field through a surface with boundary, using a more convenient surface with the same boundary.

- Statement of the two-dimensional version of the divergence theorem, and comparison with Green's theorem.