

Math 53 Final, 5/18/07, 12:30 PM – 3:30 PM

No calculators or notes. Each question is worth 10 points. Please write your solution to each of the 10 questions on a separate sheet with your name, SID#, and GSI on it. (If you are removing an incomplete for professor X, write "Incomplete/X" on each page. For Math 49, do questions 6-10 only.) To get full credit for a question, you must obtain the correct answer, put a box around it, and show correct work/justification. Please do not leave the exam between 3:00 and 3:30. Good luck!

1. Find the point on the sphere  $x^2 + y^2 + z^2 = 1$  that minimizes the function

$$f(x, y, z) = (x - 2)^2 + (y - 2)^2 + (z - 1)^2.$$

2. Find the volume of the region consisting of all points that are inside the sphere  $x^2 + y^2 + z^2 = 4$ , above the plane  $z = 0$ , and below the plane  $z = x$ .

3. Let  $R$  be the triangle with vertices  $(0, 0)$ ,  $(1, 0)$ , and  $(1/2, 1/2)$ . Evaluate the integral

$$\iint_R \frac{e^{x+y}}{x+y} dA.$$

*Hint:* Use the change of variables  $u = x + y$ ,  $v = x - y$ .

4. A particle moves along the intersection of the surfaces

$$x^2 + y^2 + 2z^2 = 4, \quad z = xy.$$

Let  $\langle x(t), y(t), z(t) \rangle$  denote the location of the particle at time  $t$ . Suppose that  $\langle x(0), y(0), z(0) \rangle = \langle 1, 1, 1 \rangle$  and  $x'(0) = 1$ . Calculate  $y'(0)$  and  $z'(0)$ .

5. Suppose  $f$  is a function on  $\mathbb{R}^2$  satisfying the following conditions on its directional derivatives:

$$D_{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle} f(x, y) = \sqrt{2}x, \quad D_{\langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle} f(x, y) = \sqrt{2}y.$$

- (a) Find  $f_x(x, y)$  and  $f_y(x, y)$ .  
(b) Assuming also that  $f(0, 0) = 0$ , find  $f(x, y)$ .

6. Let  $S$  be the triangle with vertices  $(0, 0, 0)$ ,  $(1, 0, 1)$ , and  $(1, 1, 2)$ , oriented upward. Calculate the surface integral

$$\iint_S \langle 3, 4, 5 \rangle \cdot d\mathbf{S}.$$

7. Consider the vector field

$$\mathbf{F} = \sqrt{x^2 + y^2 + z^2} \langle x, y, z \rangle.$$

- (a) There is a constant  $c$  such that  $\operatorname{div} \mathbf{F} = c\sqrt{x^2 + y^2 + z^2}$ . Find  $c$ .  
(b) Compute the outward flux of the vector field  $\mathbf{F}$  through the boundary of the solid region  $x^2 + y^2 + z^2 \leq 1$ ,  $z \geq \sqrt{x^2 + y^2}$ .
8. Calculate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $C$  is the space curve  $\mathbf{r}(t) = \langle \cos t, 0, \sin t \rangle$  for  $0 \leq t \leq 2\pi$ , and

$$\mathbf{F} = \langle \sin(x^3) + z^3, \sin(y^3), \sin(z^3) - x^3 \rangle.$$

*Hint:* Use Stokes' Theorem.

9. Let  $S_1$  be the hemisphere  $x^2 + y^2 + z^2 = 1$ ,  $z \geq 0$ , oriented upward. Let

$$\mathbf{F} = \langle x + y^2 + z^2, x^2 - y + z^2, x^2 + y^2 \rangle.$$

- (a) Use the Divergence Theorem to show that there is an oriented surface  $S_2$  in the  $x, y$  plane such that  $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}$ .  
(b) Use part (a) to calculate  $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$ .
10. Let  $C$  be the semicircle  $x^2 + y^2 = 1$ ,  $y \geq 0$ , oriented counterclockwise. Calculate the line integral

$$\int_C (-y + \cos x) dx + (x + \sin y) dy.$$

*Hint:* Part of this integral can be evaluated directly from the definition, and the rest using the Fundamental Theorem of Line Integrals.