Name: ______________________

Linear Algebra and Differential Equations

Final

Robert F. Coleman

Do problem 1 and eight of the remaining nine problems. Put your answers, as well as your name, on the exam. Problem 1 is worth 40 points, problems 2-10 are each worth 20 points. Mark below, each problem you want graded. You may use any and only those results discussed in class or in the text.

_ 1 ___

_ 2 ___

_ 3 ___

_ 4 ___

_ 5 ___

_ 6 ___

_ 7 ___

_ 8 ___

_ 9 ___

_ 10 ___

Total _____

1. True/False If false, give a counterexample. If true just say true.

(i) Suppose M is an $n \times n$ matrix such that $\det(M) \neq 0$. Then the rows of M form a basis of $\mathbf{R}^n$.

(ii) The vectors in $\mathbf{R}^3$, $(1,1,1),(1,2,4),(1,3,9),(1,4,16)$ are independant.

(iii) There are infinitely many solutions (x, y) to the system of equations: $2x+y=3$, $-3x+y=-2$ and $4x+y=5$.

(iv) If M and P are $n \times n$ matrices and P is invertible then $\operatorname{rank}(M) = \operatorname{rank}(MP)$.

(v) If the characteristic polynomial of a square matrix has multiple roots, the matrix is not diagonalizable.

(vi) If A and B commute and v is an eigenvector for both A and B, then v is an eigenvector for AB.

(vii) If A is invertible, then AB is invertible if and only if $\det(B) \neq 0$.

(viii) If A has an orthonormal eigenbasis, then A is symmetric.

(ix) Any set of three pairwise independant vectors in $\mathbf{R}^3$ is a basis for $\mathbf{R}^3$.

(x) If a real matrix M has no real eigenvalues, then the equation $v' = Mv$ has no real valued solutions.

(xi) Suppose V is an n dimensional vector space with an inner product. Then the maximal number of distinct pairwise orthogonal subspaces of V is n.

(xii) The determinant of an $n \times n$ ($n > 0$) square matrix M is zero if and only if one of the rows of M is a linear combination of the other rows.

(xiii) The product of two diagonalizable matrices is diagonalizable.

(xiv) The sum of two projections is a projection.

(xv) Similar matrices have the same eigenvalues.

(xvi) If M is a square matrix in row-echelon form with a 1 in each row, then M is invertible.

(xvii) If p and q are continuous functions on $[0, 1]$, the set of functions $f \in \mathcal{C}^2(I)$ such that $f'' + pf' + qf = 0$ is a vector space.

(xviii) If M is a constant 2×2 matrix with no real eigenvalues, the solutions of $v' = Mv$ are bounded.

(xix) If y_1 and y_2 consitute a pair of fundamental solutions of $y'' + p(x)y' + q(x)y = 0$ on I, they can have no common zeroes.

(xx) The operator on differentiable functions $f \mapsto (f')^2$ is linear.

2. Let A be an $n \times n$ matrix. Define a linear map d_A from $\mathbf{M}_n$ to itself as follows:

$$d_A(X) = AX - XA.$$

Show if A is symmetric the rank of d_A is at most $n^2 - n$.

Solution. $\dim \mathbf{M}_n = n^2$ so by the rank plus nullity theorem it suffices to show $\dim \ker d_A \geq n$. First note if A and X are diagonal, $X \in \ker d_A$. Also, the set D of diagonal matrices is a subspace of $\mathbf{M}_n$ of dimension n which does it in this case.
If A is symmetric there exists an invertible matrix M so that $M^{-1}AM$ is diagonal and so

$$0 = (M^{-1}AM)X - X(M^{-1}AM) = M^{-1}(A(MXM^{-1}) - (MXM^{-1})A)M.$$

Thus $A(MXM^{-1}) - (MXM^{-1})A = 0$ and $MAM^{-1} = 0$. Since MDM^{-1} is a subspace of dimennsion n, we get what we want.

3. Show if Q is a quadratic form on $\mathbf{R}^n$, $Q(v) = \langle v, v \rangle$ for an inner product $\langle\ ,\ \rangle$ if and only if the eigenvalues of the corresponding matrix are positive.

Solution. Let M_Q be the matrix corresponnding to Q so that

$$Q(v) = M_Q v \cdot v.$$

We know that there is an orthonormal M_Q-eigenbasis $w_1, \ldots, w_n$ of $\mathbf{R}^n$. Suppose $M_Q w_i = \lambda_i w_i$, $\lambda_i \in \mathbf{R}$. Then,

$$Q(a_1 w_1 + \cdots + a_n w_n) = \lambda_1 a_1^2 + \cdots + \lambda_n a_n^2.$$

Thus if $\langle v, w \rangle = M_Q v \cdot w$, $\langle v, v \rangle > 0$ for all $v_{\neq 0}$ if and only if all the λ_i are positive.

4. Determine whether or not

$$\begin{pmatrix} 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ -1 & 2 & 1 & 3 \\ 3 & 1 & 4 & 5 \end{pmatrix}$$

is invertible. If so compute the inverse, if not, compute the rank.

5. Suppose $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$ and

$$M = B \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} C.$$

Compute BC and use this to find the solutions $v(T)$ of the equation

$$v' = Mv$$

such that $v(0) = (1, 0, 0)^T$.

6. Let A be a 2×2 matrix. Suppose $A^n = 0$ for some integer n, show $A^2 = 0$.

7.. Solve the heat equations on $[0, 1]$;

$$(i) \qquad \begin{array}{c} u_t = u_{xx} \\ u(0,t) = 1 \quad u(1,t) = 0 \\ u(x,0) = x \end{array} \qquad \text{and} \qquad (\text{ii}) \quad \begin{array}{c} w_t = w_{xx} \\ w(0,t) = 0 \quad w(1,t) = 1 \ . \\ w(x,0) = x. \end{array}$$

What equation with what initial conditions does $au + bw$ satisfy where a and b are scalars.

8. Let A be a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Suppose that $|\lambda_1| < 1$ for every i. Prove that as $m \mapsto \infty$, all the entries of A^m approach zero.

9. Find a 2×2 constant matrix M over $\mathbf{R}$ such that

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-x}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{4x} \right\}$$

is a set of fundamental solutions of the equation $v' = Mv$.

10. Suppose A and B are commuting $n \times n$ matrices. (a) Show that if v is an eigenvector of B with eigenvalue b, so is Av. (b) Show that if B is diagonalizable and the diagonal entries are distinct then A is diagonalizable also.

Solution.

(a)

$$B(Av) = (BA)v = (AB)v = A(Bv) = A(bv) = b(Av)$$

(b) The hypotheses imply, there is a basis $v_1, \ldots, v_n$ of $\mathbf{R}^n$ such that $Bv_i = \lambda_i v_i$ and $\lambda_i \neq \lambda_j$ if $i \neq j$. It follows, in particular, that the set V_i of vectors v such that $Bv = \lambda_i v$ is a one dimensional subspace of $\mathbf{R}^n$ spanned by v_i.

It follows from (a) that $Av_i \in V_i$ and so there exist scalars $\mu_i \in \mathbf{R}$ so that $Av_i = \mu_i v_i$. So the matrix of A with respect to this basis is

$$\begin{pmatrix} \mu_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_n \end{pmatrix}$$