Solutions to Midterm II

Problem 1: (20pts)

Find the most general solution $u(x, y)$ of the following equation consistent with the boundary condition stated

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 3x\,, \qquad u = x^2 \text{ on the line } y = 0 \qquad (1)$$

Solution 1: Since the system (1) is linear, the solution is given as a superposition of a homogenous and a particular solution

$$u(x, y) = u_H(x, y) + u_P(x, y)$$

We start by computing the homogenous solution;

Characteristic Equation:

$$\frac{dy}{dx} = -\frac{x}{y}$$

Characteristic Curves:

$$C = x^2 + y^2$$

Homogenous solution:

$$u_H(x, y) = f\left(x^2 + y^2\right)$$

Next, we solve for a particular solution to (1)

We assume

$$u_P(x, y) = ax + by$$

If we substitute the particular solution into (1),
We obtain

$$ay - xb = 3x \qquad \Rightarrow a = 0, b = -3$$

Hence,

$$u_P(x, y) = -3y$$

General solution:

$$u(x, y) = f\left(x^2 + y^2\right) - 3y$$

On $y = 0$:

$$u(x,0) = f\left(x^2\right) = x^2$$

Hence,

$$u(x, y) = x^2 + y^2 - 3y$$

Problem 2: (30pts)

Solve for $u(x,y)$, $$6\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

Subject to $u = 2x+1$ and $\frac{\partial u}{\partial y} = 4-6x$ both on the line $y=0$

Solution 2:

Characteristic equations:

$$\frac{dy}{dx} = \frac{-5 \pm \sqrt{(-5)^2 - 4(6)(1)}}{2(6)} = \frac{-5 \pm 1}{12}$$

Characteristic Curves:

$$x + 3y = C_1 \qquad \text{and} \qquad x + 2y = C_2$$

General solution:

$$u(x,y) = f(x+3y) + g(x+2y)$$

Boundary conditions:

$$u(x,0) = f(x) + g(x) = 2x+1 \qquad \text{and} \qquad u_y(x,0) = 3f'(x) + 2g'(x) = 4-6x \qquad (1,2)$$

Differentiate (1),

$$f'(x) + g'(x) = 2 \qquad\qquad 3f'(x) + 2g'(x) = 4-6x \qquad (3,4)$$

We solve equations (3) and (4) simultaneously

$$f'(x) = -6x \qquad\qquad g'(x) = 2+6x \qquad (5,6)$$

Integrate (5,6)

$$f(x) = k_o - 3x^2 \qquad\qquad g(x) = k_1 + 2x + 3x^2$$

So that

$$f(x+3y) = k_o - 3(x+3y)^2 \qquad\qquad g(x+2y) = k_1 + 2(x+2y) + 3(x+2y)^2$$

Hence,

$$u(x,y) = f(x+3y) + g(x+2y) = k_o - 3(x+3y)^2 + k_1 + 2(x+2y) + 3(x+2y)^2$$

Simplify

$$u(x,y) = 2x + 4y - 6xy - 15y^2 + k_o + k_1$$

Boundary condition

$$u(x,0) = 2x + (k_o + k_1) = 2x+1, \qquad \Rightarrow k_o + k_1 = 1$$

So that

$$u(x,y) = 2x + 4y - 6xy - 15y^2 + 1$$

<u>Problem 3</u>: (5pts)

Show that there is no solution of

$$\nabla^2 u = f \quad \text{in} \quad D \qquad\qquad \frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial D \qquad\qquad (1,2)$$

In three dimensions, unless

$$\iiint_D f dV = \iint_{\partial D} g dS \qquad\qquad (3)$$

Hints: Use the divergence theorem $\iiint_D \nabla \cdot \vec{F} dV = \iint_{\partial D} \vec{F} \cdot \hat{n} dS$

<u>Solution 3</u>:

We integrate (1)

$$\iiint_D f dV = \iiint_D (\nabla \cdot \nabla u) dV \qquad\qquad (4)$$

We now apply the divergence theorem on the right side of (4)

$$\iiint_D f dV = \iiint_D (\nabla \cdot \nabla u) dV = \iint_{\partial D} \nabla u \cdot \hat{n} dS = \iint_{\partial D} \frac{\partial u}{\partial n} dS = \iint_{\partial D} g dS \qquad \text{Q.E.D}$$

<u>Problem 4</u>: (45pts)

A thin sheet of metal coincides with a unit square in the $xy-plane$. Initially, the temperature in the sheet is $T(x,y,0) = h(x,y)$. If there are no sources of heat in the sheet, find the temperature at any point at any subsequent time, given that the right and left faces of the sheet are insulated, the temperature at the lower edge is maintained at zero, and the temperature at the upper edge is prescribed by $T(x,1,t) = T_0 \cos(\pi x)$.

<u>Solution 4</u>:

The governing equation to be solved is

$$\frac{1}{\alpha^2} \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \qquad \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1 \end{array} \qquad (1)$$

$$T_x(0,y,t) = 0, \quad T_x(1,y,t) = 0, \quad T(x,0,t) = 0, \quad T(x,1,t) = T_0 \cos(\pi x), \quad T(x,y,0) = h(x,y)$$

We define our solution as the sum of a steady and a transient component

$$T(x,y,t) = \Theta(x,y) + \Phi(x,y,t) \qquad\qquad (2)$$

We now substitute (2) into (1) and rearrange

$$\frac{1}{\alpha^2}\frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \left[\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2}\right] \qquad \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1 \end{array} \tag{3}$$

$$\Theta_x(0,y) + \Phi_x(0,y,t) = 0, \quad \Theta_x(1,y) + \Phi_x(1,y,t) = 0, \quad \Theta(x,0) + \Phi(x,0,t) = 0$$

$$\Theta(x,1) + \Phi(x,1,t) = T_o \cos \pi x, \quad \Theta(x,y) + \Phi(x,y,0) = h(x,y)$$

We now make the following choices for $\Phi(x,y,t)$

$$\frac{1}{\alpha^2}\frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \qquad \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1 \end{array} \tag{4a}$$

$$\Phi_x(0,y,t) = 0, \quad \Phi_x(1,y,t) = 0, \quad \Phi(x,0,t) = 0, \quad \Phi(x,1,t) = 0, \quad \Phi(x,y,0) = g(x,y)$$

And for $\Theta(x,y)$

$$\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} = 0, \qquad \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1 \end{array} \tag{4b}$$

$$\Theta_x(0,y) = 0, \quad \Theta_x(1,y) = 0, \quad \Theta(x,0) = 0, \quad \Theta(x,1) = T_0 \cos(\pi x)$$

We solve (4a) by assuming separable solutions of the form,

$$\Phi(x,y,t) = X(x)Y(y)T(t) \tag{5}$$

We now substitute (5) into (4a)

$$\frac{1}{\alpha^2} XYT' = X''YT + XY''T$$

We now divide through by $\Phi(x,y,t) = X(x)Y(y)T(t)$

$$\frac{1}{\alpha^2}\frac{T'}{T} = \frac{X''}{X} + \frac{Y''}{Y} \tag{6}$$

The left side of (6) is a function of t only. The first term on the right is a function of x only and the last term a function of y only. The only way for both sides to be equal is if they are each equal to some constant say $-\lambda^2, -\mu^2$.

$$\frac{X''}{X} = -\lambda^2, \qquad \frac{Y''}{Y} = -\mu^2, \qquad \frac{1}{\alpha^2}\frac{T'}{T} = -\left(\lambda^2 + \mu^2\right)$$

By inspection, we have chosen the constants to be negative. So that

$$X''(x) + \lambda^2 X(x) = 0, \qquad Y''(y) + \mu^2 Y(y) = 0, \qquad T'(t) + \alpha^2\left(\lambda^2 + \mu^2\right)T(t) = 0 \tag{7a}$$

For non-trivial solutions, we select our boundary conditions as

$$X'(0) = X'(1) = Y(0) = Y(1) = 0 \tag{7b}$$

We now solve (7)

$$X(x) = c_1 \cos(\lambda x) + c_2 \sin(\lambda x), \quad Y(y) = c_3 \cos(\mu y) + c_4 \sin(\mu y), \quad T(t) = c_5 \exp\left(-\alpha^2\left(\lambda^2 + \mu^2\right)t\right)$$

We now apply the homogenous boundary conditions

$$X'(0) = \lambda c_2 = 0, \qquad X'(1) = -c_1 \lambda \sin(\lambda) = 0$$

For nontrivial solutions, $c_2 = 0$, $\lambda_n = n\pi$ $n = 1,2,........$

So that, $$X_n(x) = c_n \cos(n\pi x) \tag{8}$$

$$Y(0) = c_3 = 0, \qquad Y(1) = c_4 \sin(\mu) = 0$$

For nontrivial solutions, $\mu_m = m\pi$ $m = 1,2,........$

So that $$Y_m(y) = b_m \sin(m\pi y)$$

Hence, $$T_{mn}(t) = a_{mn} \exp\left(-\alpha^2\pi^2\left(n^2 + m^2\right)t\right)$$

The solutions

$$\Phi_{mn}(x, y, t) = \gamma_{mn} \cos(n\pi x)\sin(m\pi y)\exp\left[-\alpha^2\pi^2\left(n^2 + m^2\right)t\right]$$

Since the system is linear, a complete solution is given as a linear combination of solutions

$$\Phi(x, y, t) = \sum_{n}^{\infty}\sum_{m}^{\infty}\gamma_{mn} \cos(n\pi x)\sin(m\pi y)\exp\left[-\alpha^2\pi^2\left(n^2 + m^2\right)t\right]$$

The constants A_{mn} are given as

$$A_{mn} = \int_0^1\int_0^1 g(x, y)\sin(m\pi y)\cos(n\pi x)dxdy$$

We now look at solutions to (4b). We assume separable solutions of the form $\Theta(x, y) = G(x)H(y)$ so that

$$G''(x) + \lambda^2 G(x) = 0 \quad \text{and} \quad H''(y) - \lambda^2 H(y) = 0 \tag{9a,b}$$

$$X'(0) = X'(1) = Y(0) = 0$$

We already solved (9a). The solutions are given in (8). So we look for the solutions of (9b)

$$Y(y) = c_3 \exp(\lambda y) + c_4 \exp(-\lambda y)$$

$$Y(0) = c_3 + c_4 = 0, \qquad \Rightarrow Y_n(y) = \begin{cases} y & n = 0 \\ b_n \sinh(n\pi y) & n = 1,2,.......... \end{cases}$$

The solutions

$$\Theta_n(x,y) = \gamma_o y + \gamma_n \cos(n\pi x)\sinh(n\pi y) \qquad n = 1,2,........$$

Since the system is linear, a complete solution is given as a linear combination of solutions

$$\Theta(x,y) = \sum_{n=1}^{\infty} \Theta_n(x,y) = \gamma_o y + \sum_{n=1}^{\infty} \gamma_n \cos(n\pi x)\sinh(n\pi y)$$

At this point, we apply the non homogenous boundary condition

$$\Theta(x,1) = T_o \cos \pi x = \gamma_o + \sum_{n=1}^{\infty} \gamma_n \cos(n\pi x)\sinh(n\pi)$$

By comparing coefficients on the two sides of the above equation, we see that only the $n = 1$ term survives, yielding

$$\gamma_1 = \frac{T_o}{\sinh(\pi)}$$

So that

$$\Theta(x,y) = \frac{T_0}{\sinh \pi} \cos(\pi x)\sinh(\pi y)$$

Finally,

$$T(x,y,t) = \frac{T_o}{\sinh \pi} \cos(\pi x)\sinh(\pi y) + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \gamma_{mn} \sin(m\pi y)\cos(n\pi x)\exp\left[-\alpha^2 \left(m^2 + n^2\right)\pi^2 t\right]$$