1 Math H104 Midterm 2

Professor J. Sussan. Fall 2007.
Each problem is worth 20 points. You have 50 minutes to complete the exam. Please show your work; clarity of exposition counts!

1.1

Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces. Let $f : X \to Y$ be uniformly continuous and let $g : Im(f) \to Z$ be uniformly continuous. where $\mathrm{Im}(f)$ is the image of f. Prove that $g \circ f : X \to Z$ is uniformly continuous.

1.2

Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that $f'(x) \neq 1$ for any $x \in [a, b]$. Prove that there exists at most one $c \in [a, b]$ such that $f(c) = c$.

1.3

Suppose $f : [0, 1] \to [0, 1]$ is a continuous function such that $f(0) = 1, f(1) = 0$. Prove there exists $c \in [0, 1]$ such that $f(c) = c$.

1.4

Let (a_n) be a bounded sequence of real numbers such that $\sum_{n=1}^{\infty} a_n$ diverges. Prove that $sum_{n=1}^{\infty} \frac{a_n}{a_n+1}$ diverges as well.

1.5

Let (X, d) be a complete metric space and let $f : X \to X$ be such that $d(f(x), f(y)) \leq \frac{1}{2} d(x, y)$ for all $x, y \in X$. Prove that there exists $c \in X$ such that $f(c) = c$.