Mathematics 1B. Fall Semester 2006 **Professor: Daniel Tataru**

Midterm 2 Solutions

(20) 1. Determine the interval of convergence of the following series. Do they converge at endpoints ?

$$a) \qquad \sum_{n=1}^{\infty} \frac{(x-1)^{2n}}{\sqrt{n}\, 4^n}$$

Solution: Using the ratio test we compute

$$\lim_{n\to\infty} \frac{\dfrac{(x-1)^{2(n+1)}}{\sqrt{n+1}\, 4^{n+1}}}{\dfrac{(x-1)^{2n}}{\sqrt{n}\, 4^n}} = \lim_{n\to\infty} \frac{(x-1)^2}{4} \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{(x-1)^2}{4}$$

The limit is less than 1 if $|x-1| < 2$. Hence the radius of convergence is $R = 2$. At the endpoints we have $x - 1 = \pm 2$ and the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

which is a divergent p-series. Thus the interval of convergence is $(-1, 3)$.

$$b) \qquad \sum_{n=2}^{\infty} \ln\left(\frac{n+1}{n-1}\right) x^n$$

Solution: Using the Taylor expansion for $\ln(1+x)$ we write

$$\ln\left(\frac{n+1}{n-1}\right) = \ln\left(1 + \frac{2}{n-1}\right) = \frac{2}{n-1} - \frac{2}{(n-1)^2} + \cdots$$

Then for the ratio test we compute

$$\lim_{n\to\infty} \frac{\ln\left(\dfrac{n+2}{n}\right) x^{n+1}}{\ln\left(\dfrac{n+1}{n-1}\right) x^n} = x \lim_{n\to\infty} \frac{\dfrac{2}{n} - \dfrac{2}{n^2} + \cdots}{\dfrac{2}{n-1} - \dfrac{2}{(n-1)^2} + \cdots} = x$$

The limit is less than 1 if $|x| < 1$. Hence the radius of convergence is $R = 1$. At the endpoint $x = -1$ we obtain the alternating series

$$\sum_{n=2}^{\infty} (-1)^n \ln\left(1 + \frac{2}{n-1}\right)$$

Due to the expansion above we have $\ln\left(\dfrac{n+1}{n-1}\right) \searrow 0$ as $n \to \infty$ therefore the series converges by the alternating test.

At the endpoint $x = 1$ we obtain the series

$$\sum_{n=2}^{\infty} \ln\left(1 + \frac{2}{n-1}\right)$$

Due to the expansion above this is comparable to the harmonic series $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n-1}$ which diverges.

Thus the interval of convergence is $[-1, 1)$.

(20) 2. Find the Maclaurin series expansion of the following functions. Determine where the expansions are valid (i.e. for what values of x they converge).

$$a) \qquad f(x) = \frac{x}{x^2 + x - 2}$$

Solution: Using partial fractions we write

$$f(x) = \frac{x}{x^2 + x - 2} = \frac{x}{(x+2)(x-1)} = \frac{2}{3(x+2)} + \frac{1}{3(x-1)} = \frac{1}{3}\left(\frac{1}{1+\frac{x}{2}} - \frac{1}{1-x}\right)$$

Then using the geometric series we write

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad \frac{1}{1+\frac{x}{2}} = \sum_{n=0}^{\infty} (-1)^n 2^{-n} x^n$$

Summing up we obtain

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{3}\left((-1)^n 2^{-n} - 1\right) x^n$$

The radius of convergence is 1 for the first term and 2 for the second, so after adding them up we obtain $R = 1$. At the endpoints $x = \pm 1$ the series diverges since the general term does not go to 0. Hence the interval of convergence is $(-1, 1)$.

$$b) \qquad f(x) = \sqrt{1 + x^2}$$

Solution: We use the binomial series

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n = \sum_{n=0}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} x^n = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdots (2n-3)}{2^n n!} x^n$$

and replace x by x^2 to obtain

$$\sqrt{1+x^2} = 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdots (2n-3)}{2^n n!} x^{2n}$$

The binomial series converges for $|x| < 1$ therefore our series also converges for $|x| < 1$. This can also be verified directly using the ratio test. At the endpoints $x = \pm 1$ we obtain the alternating series

$$\sum_{n=0}^{\infty} (-1)^{n-1} a_n, \qquad a_n = \frac{1 \cdot 3 \cdots (2n-3)}{2^n n!}$$

It is easily verified that the sequence a_n is decreasing, but harder to show that it converges to 0. We have

$$a_n = \frac{1}{2}\frac{3}{4} \cdots \frac{2n-3}{2n-2} \cdot \frac{1}{2n} < \frac{1}{2n} \to 0$$

This implies that $a_n \to 0$. Then the interval of convergence is $[-1, 1]$.

(20) 3. a) Find the third order Taylor polynomial of $\tan x$ at $\pi/4$.
Solution: For $f(x) = \tan x$ we compute

$$f'(x) = \sec^2 x, \qquad f''(x) = 2\sec^2 x \tan x, \qquad f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$$

We evaluate them at $\pi/4$ using $\tan \pi/4 = 1$, $\sec \pi/4 = \sqrt{2}$. This gives

$$f(\pi/4) = 1, \qquad f'(\pi/4) = 2, \qquad f''(\pi/4) = 4, \qquad f'''(\pi/4) = 16$$

Then the third order Taylor polynomial of $\tan x$ at $\pi/4$ is

$$P_3(x) = 1 + 2(x - \pi/4) + 2(x - \pi/4)^2 + \frac{8}{3}(x - \pi/4)^3$$

b) Find the Maclaurin series for a function f which solves the differential equation

$$f''(x) = xf(x), \qquad f(0) = 1, \ f'(0) = 0$$

What is the radius of convergence ?
Solution: If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ therefore

$$f'(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = 1 \cdot 2a_2 + 2 \cdot 3a_3 x + 3 \cdot 4a_4 x^2 + \cdots + (n+2)(n+1)a_{n+2}x^n + \cdots$$

On the other hand

$$xf(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = a_0 x + a_1 x^2 + \cdots + a_{n-1}x^n + \cdots$$

Identifying the coefficients in the two power series we obtain $a_2 = 0$ and

$$(n+2)(n+1)a_{n+2} = a_{n-1}, \qquad n \geq 1$$

From the initial data we also know that $a_0 = 1$, $a_1 = 0$. Then we can iteratively compute the coefficients a_n (e.g. we use the above formula with $n = 1$ to compute a_3, etc.):

$$1, 0, 0, \frac{1}{2 \cdot 3}, 0, 0, \frac{1}{2 \cdot 3 \cdot 5 \cdot 6}, 0, 0, \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}, \cdots$$

This gives the Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)3n} x^{3n}$$

To compute the radius of convergence we use the ratio test. We have

$$\lim_{n\to\infty} \frac{\dfrac{x^{3n+3}}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)3n(3n+2)(3n+3)}}{\dfrac{x^{3n}}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)3n}} = \lim_{n\to\infty} \frac{x^3}{(3n+2)(3n+3)} = 0$$

Hence the series converges for all x.

(20) 4. Sketch the direction field of

$$y' = y^3 - y$$

and determine the equilibrium solutions. Are they stable ?
Solution: a) We check the sign of y':

y		-1		0		1	
y'	$-$	0	$+$	0	$-$	0	$+$

The equilibrium solutions are $y = \pm 1$ and $y = 0$.
b) We sketch the direction field (see the picture in problem 1b, Section 9.2 **but with the x axis reversed**)
c) Sketch a few solutions which follow the direction field. The solution $y = 0$ is stable, but $y = \pm 1$ are not.

(20) 5. Solve the initial value problems

$$a) \qquad \frac{dx}{dt} = 2t(1+x^2), \qquad x(0) = 0$$

Solution: This is a separable equation. We compute

$$\frac{dx}{1+x^2} = 2tdt, \qquad \int \frac{dx}{1+x^2} = \int 2tdt + C$$

which gives

$$\tan^{-1} x = t^2 + C$$

Using the initial data we obtain $C = 0$, therefore the solution is

$$x(t) = \tan t^2$$

$$b) \qquad \frac{dx}{dt} = x + \sin t, \qquad x(0) = 1$$

This is a linear equation, which we rewrite as

$$x' - x = \sin t$$

The integrating factor is e^{-t}. Multiplying by it in both sides gives

$$e^{-t}x' - e^{-t}x = e^{-t}\sin t \qquad \Leftrightarrow \qquad (e^{-t}x)' = e^{-t}\sin t$$

Hence integrating by parts we obtain

$$e^{-t}x(t) = \int e^{-t}\sin t dt = -\frac{1}{2}e^{-t}(\sin t + \cos t) + C$$

so the general solution is

$$x(t) = -\frac{1}{2}(\sin t + \cos t) + Ce^t$$

Using the initial data in this equation gives $C = \frac{3}{2}$, therefore

$$x(t) = -\frac{1}{2}(\sin t + \cos t) + \frac{3}{2}e^t$$