Midterm #2, Rezakhanlou Math 53. FA 06

1. *(24 points)* (a) Find the equation of the tangent plane to the surface

$$x^{1/3} + y^{1/3} + z^{1/3} = 1$$

at (x_0, y_0, z_0). Show that the sum of the square root of the $x-$, $y-$, and $z-$intercepts of any tangent plane is 1.

$\nabla F = \frac{1}{3} < x_0^{-2/3}, y_0^{-2/3}, z_0^{-2/3} >$,

Tangent plane: $x_0^{-2/3}(x-x_0) + y_0^{-2/3}(y-y_0) + z_0^{-2/3}(z-z_0) = 0$,

$x_0^{-2/3}x + y_0^{-2/3}y + z_0^{-2/3}z = 1$

Intercepts: $x_0^{2/3}, y_0^{2/3}, z_0^{2/3}$

So $\sqrt{x_0^{2/3}} + \sqrt{y_0^{2/3}} + \sqrt{z_0^{2/3}} = 1$

(b) Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex on the surface $x^{1/3} + y^{1/3} + z^{1/3} = 1$.

Let $f(x,y,z) = xyz$, $F(x,y,z) = 1$,

$\nabla f = <yz, xz, xy> = \lambda \nabla F = \frac{\lambda}{3} < x^{-2/3}, y^{-2/3}, z^{-2/3} >$

$\frac{\lambda}{3} = x^{2/3}yz = xzy^{2/3} = xyz^{2/3}$. Hence $x^{1/3} = y^{1/3} = z^{1/3}$ or $x=y=z$.

Answer: $x = y = z = 3^{-3}$, volume $= 3^{-9}$.

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(c) Assume that $x^{1/3} + y^{1/3} + z^{1/3} = 1$. Find the partial derivatives of z with respect to x and y. Approximate the value of z when $x = 1.01$ and $y = .97$.

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\left(\frac{x}{z}\right)^{-2/3}$, $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\left(\frac{y}{z}\right)^{-2/3}$.

If $x_0 = y_0 = 1$, then $z_0 = -1$. Now

$z(x,y) \approx z(x_0,y_0) + \frac{\partial z}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial z}{\partial y}(x_0,y_0)(y-y_0)$

$z(1.01, .97) \approx -1 + (-1)(.01) + (-1)(-.03) = -.98$

2. *(10 points)* (a) Let $f(x, y, z)$ be a function of three variables and let

$$g(r, s) = f(r - s, r + s, se^r).$$

Find g_{rs}.

$$g_r = f_x(r-s,\ r+s,\ se^r) + f_y(r-s,\ r+s,\ se^r) + f_z(r-s, r+s, se^r)\, se^r$$

$$g_{rs} = -f_{xx} + f_{xy} + f_{xz}\, e^r$$

$$- f_{yx} + f_{yy} + f_{yz}\, e^r$$

$$+ (-f_{zx} + f_{zy} + f_{zz}\, e^r)\, se^r + f_z\, e^r.$$

3. *(18 points)* (a)Find and classify the critical points of $f(x, y) = x^3 - 3xy + y^3$.

$$\nabla f = \langle 3x^2 - 3y \; , -3x + 3y^2 \rangle = \langle 0, 0 \rangle \Rightarrow \quad x^2 = y \, , \, y^2 = x .$$

So, $x, y \geq 0$ and $x^4 = x$. Or $x(x-1)(x^2+x+1) = 0$. Hence $x = 0, 1$.

$f_{xx} = 6x$, $f_{xy} = -3$, $f_{yy} = 6y$, $D = f_{xx} f_{yy} - f_{xy}^2 = 36xy - 9$.

Thus $\begin{cases} (0,0) & \text{is a saddle} \quad (D < 0) \\ (1,1) & \text{is a min} \quad (D > 0, \; f_{xx} > 0 .) \end{cases}$

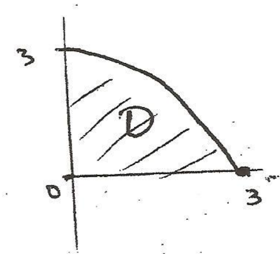
(b) Find the maximum and the minimum of f on the set $D = \{(x, y) : x \geq 0, y \geq 0, x^3 + y^3 \leq 27\}$.

Inside D : The only critical value is $f(1,1) = -1$

Boundary line $y=0$: Max $f = 27$, Min $f = 0$

Boundary line $x=0$: Max $f = 27$, Min $f = 0$

Boundary curve $x^3 + y^3 = 27$:



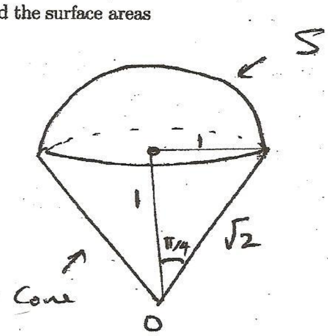
$$7 - 3.3.3 \leq f = 27 - 3xy \leq 27$$

$$0 \leq f \leq 27 .$$

Thus Max of f on D $= 27$

Min of f on D $= -1$

4. *(12 points)*
Let S be a spherical cap formed by cutting the sphere $x^2+y^2+z^2=2$ with a cone having vertex angle $\pi/4$ with vertex at the center of the sphere. Find the surface areas of the cap S and the cone.



For $z = \sqrt{2-x^2-y^2}$, $w = \sqrt{x^2+y^2}$

$x^2+y^2 \le 1$ we have

$$\text{Area of } S = \iint_{x^2+y^2\le 1} \sqrt{1+z_x^2+z_y^2}\,dx\,dy$$

$$= \iint_{x^2+y^2\le 1} \left(1+\frac{x^2+y^2}{2-x^2-y^2}\right)^{1/2} dx\,dy$$

$$= \iint_{x^2+y^2\le 1} \frac{\sqrt{2}}{\sqrt{2-x^2-y^2}}\,dx\,dy = \sqrt{2}\cdot 2\pi \int_0^1 \left(\sqrt{2-r^2}\right)^{-1} r\,dr$$

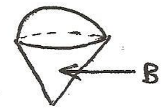
$$= 2\sqrt{2}\,\pi \left[-\sqrt{2-r^2}\right]_0^1 = 2\sqrt{2}\,\pi\,(\sqrt{2}-1)$$

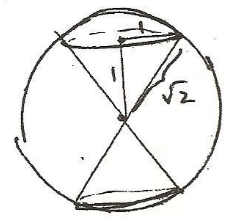
$$\text{Area of Cone} = \iint_{x^2+y^2\le 1} \sqrt{1+w_x^2+w_y^2}\,dx\,dy = \iint_{x^2+y^2\le 1} \sqrt{1+\frac{x^2+y^2}{x^2+y^2}}\,dx\,dy$$

$$= \sqrt{2}\ \text{area of } (x^2+y^2\le 1) = \sqrt{2}\,\pi.$$

5. *(11 points)* Let A be a part of the cone $x^2+y^2 \le z^2$ inside the sphere $x^2+y^2+z^2=2$. What is the mass of A if the mass density is $\rho(x,y,z) = |z|$.

$$\text{Mass} = 2 \iiint_B z \, dx\,dy\,dz.$$





$$\text{Mass} = 2 \iint_{x^2+y^2\le 1} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z \, dz$$

$$= \iint_{x^2+y^2\le 1} \left[(2-x^2-y^2) - (x^2+y^2)\right] dx\,dy$$

$$= \iint_{x^2+y^2\le 1} (2-2(x^2+y^2)) \, dx\,dy$$

$$= 2\pi \int_0^1 (2-2r^2)\, r \, dr$$

$$= 2\pi \left[r^2 - \frac{r^4}{2}\right]_0^1 = \pi.$$