Math 110 Professor K. A. Ribet
Midterm Exam October 31, 2002

Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Please write your name on each sheet of paper that you turn in. Don't trust staples to keep your papers together. Explain your answers as is customary and appropriate. Your paper is your ambassador when it is graded. In this midterm, the scalar field F will be the field of real numbers unless otherwise specified.

These solutions were written by Ken Ribet.

1. *Let A be the 100×100 matrix of real numbers whose entry in the (i, j)th place is ij for $1 \le i, j \le 100$. Determine the rank of A. (Figure out what it is and show that the rank is what you say it is.)*

Looking over the papers that were handed in, I got the impression that this problem was easy for most students. As people pointed out, the jth column is j times the first column, so the span of the set of columns is the span of the first column. This span is 1-dimensional, so the rank is 1. More generally, we can take two vectors $(a_1, \dots, a_n)$ and $(b_1, \dots, b_m)$ and form the matrix whose (i, j)th entry is $a_i b_j$. This is called the *outer product* of the two vectors. The span of this outer product matrix is 1-dimensional if the vectors are both non-zero and 0-dimensional otherwise.

2. *Express* $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ *as a product of elementary matrices.*

The idea is to bring A to the identity matrix I by a sequence of elementary row operations. Each operation corresponds to left multiplication by an elementary matrix E_i. When I did this, there were three operations; I got $E_3 E_2 E_1 A = I$, where E_1 corresponds to the first operation, and so on. Then $A = E_1^{-1} E_2^{-1} E_3^{-1}$. The inverses of the E_i are again elementary matrices, so this is the desired expression. I did this problem on paper before typing the exam and got

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let me know if there are misprints or errors here. A final comment: as Tom stressed to me, there is no "best way" to get from A to I by elementary operations. Your answer may be different from mine and still be correct. In fact, few students got my answer but most students got answers that are essentially correct.

3. *Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces and let $T^{\mathrm{t}} : W^* \to V^*$ be the transpose of T. Establish the equations*

$$\operatorname{nullity} T^{\mathrm{t}} + \operatorname{rank} T = \dim W, \qquad \operatorname{nullity} T + \operatorname{rank} T^{\mathrm{t}} = \dim V.$$

Show that T is onto if and only if T^{t} is 1-1, and that T is 1-1 if and only if T^{t} is onto.

The general formula "$\operatorname{nullity} T + \operatorname{rank} T = \dim V$" becomes the second equation once we realize that the rank of T^{t} and the rank of T are equal. This equality, in matrix-speak, is the statement that the row rank and column rank of a matrix coincide. If we apply the general formula to T^{t}, we get the formula $\operatorname{nullity} T^{\mathrm{t}} + \operatorname{rank} T^{\mathrm{t}} = \dim W^*$. Now W^* and W have the same dimension. Also, T and T^{t} have the same rank (as we've seen). Hence we get the first of the two equations. To continue, we observe that T is onto if and only if its rank is the dimension of W; by the first equation, this is true if and only if the nullity of T^{t} is 0, i.e., if and only if T^{t} is 1-1. Similarly, T is 1-1 if and only if the nullity of T is 0; by the second equation, this is true if and only if $\operatorname{rank} T^{\mathrm{t}} = \dim V$. This latter equality holds precisely when T^{t} is onto.

4. *Let $V = \mathbf{P}_2(\mathbf{R})$ be the real vector space of polynomials of degree ≤ 2. You have proved that the following linear functionals form a basis for the dual space V^*: $\mathrm{f}_- =$ evaluation at -1, $\mathrm{f}_0 =$ evaluation at 0, $\mathrm{f}_+ =$ evaluation at $+1$. Let $T : V \to V^*$ be the linear transformation that maps a polynomial $p(x)$ to the functional f_p defined by the formula*

$$\mathrm{f}_p(q) = 120 \int_0^1 p(x)q(x)\,dx, \qquad q \in V.$$

Find the matrix of T with respect to the bases $\{\,1, x, x^2\,\}$ of V and $\{\,\mathrm{f}_-, \mathrm{f}_0, \mathrm{f}_+\,\}$ of V^*.

This problem is related to the most recent quiz, which concerned the fact that V^* has $\{\,\mathrm{f}_-, \mathrm{f}_0, \mathrm{f}_+\,\}$ as a basis. This means, in particular, that any linear functional f on V may be written $a\mathrm{f}_- + b\mathrm{f}_0 + c\mathrm{f}_+$. We find a, b and c by evaluating f on 1, x and x^2, three convenient elements of V. Namely, we have $f(1) = a + b + c$, $f(x) = -a + c$ and $f(x^2) = a + c$; note, for instance, that f_- takes the value -1 on x, that f_0 is 0 on x, and that f_+ is 1 on x. We have three simple equations for a, b and c; solving them, we get

$$a = \frac{f(x^2) - f(x)}{2}, \quad b = f(1) - f(x^2), \quad c = \frac{f(x) + f(x^2)}{2}.$$

We now have to tabulate the three values $f(1)$, $f(x)$, $f(x^2)$ in the three cases $f = f_1$, $f = f_x$, $f = f_{x^2}$ that correspond to the basis vectors $p = 1$, $p = x$ and $p = x^2$ of V and then plug in to find a, b and c:

f	$f(1)$	$f(x)$	$f(x^2)$	a	b	c
f_1	120	60	40	-10	80	50
f_x	60	40	30	-5	30	35
f_{x^2}	40	30	24	-3	16	27.

The matrix that we were supposed to calculate is apparently $\begin{pmatrix} -10 & -5 & -3 \\ 80 & 30 & 16 \\ 50 & 35 & 27 \end{pmatrix}$.