Math 110 Spring 2008

Final

*Write your name and SID on the front of your blue book. All answers and work should also be written in your blue book. You must **JUSTIFY** your answers, so show your work. Partial credit will be awarded even if answers are incorrect. No notes, books, or calculators. Good luck!*

1. (15 pts.) The matrix
$$A = \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}$$
has characteristic polynomial $f(t) = (t-\lambda_1)^2(t-\lambda_2)^2$, where $\lambda_1 = 1+i$ and $\lambda_2 = 1-i$.
a. (5 pts.) Fact: $A^* = -\frac{1}{2}A^3$. Use this to prove that A is normal.

SOLUTION: $A^*A = -1/2A^3A = A(-1/2A^3) = AA^*$.

b. (5 pts.) From a. it follows that A has a spectral decomposition: $A = \lambda_1 T_1 + \lambda_2 T_2$. Compute T_1 and T_2 by finding polynomials $g_i(t)$ such that $g_i(A) = T_i$. You must compute the g_i and T_i for full credit.

SOLUTION: $g_1(t) = (t-\lambda_2)/(\lambda_1-\lambda_2) = \frac{1}{2i}(t-(1+i))$ and similarly $g_2(t) = \frac{-1}{2i}(t-(1-i))$. You can now easily compute $T_i = g_i(A)$.

c. (5 pts.) Using b., find a linear polynomial $g(t) = ct + d$ such that $g(A) = A^*$.

SOLUTION: $g(t) = \overline{\lambda_1}g_1(t) + \overline{\lambda_2}g_2(t) = -t + 2$.

2. (15 pts.) The matrices
$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & -1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$
both have characteristic polynomial $f(t) = (t-2)^2(t-1)$.
a. (5 pts.) Compute Jordan canonical forms J_1 and J_2 for A and B.

SOLUTION: $J_1 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $J_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

b. (5 pts.) Determine whether A and B are similar.

SOLUTION: No, since they are different Jordan type.

c. (5 pts.) Find a Q such that $J_1 = Q^{-1}AQ$; i.e., compute a Jordan basis for A.

SOLUTION: One possible Q is $Q = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$.

3. (10 pts.)Suppose $A, B \in \mathrm{M}_{n\times n}(\mathbb{C})$ are unitarily equivalent.
a. (2 pts.) Write down what this means; i.e., give the the definition of unitarily equivalent.

SOLUTION: There exists a Q with $Q^{-1} = Q^*$ such that $B = Q^*AQ$.

b. (4 pts.) Prove that A is normal $\Leftrightarrow$ B is normal.

SOLUTION: If $A^*A = AA^*$, then $B^*B = (Q^*AQ)^*Q^*AQ = Q^*A^*QQ^*AQ = Q^*A^*AQ = Q^*AA^*Q = Q^*AQ(Q^*AQ)^* = BB^*$.

c. (4 pts.) Prove that A is self-adjoint $\Leftrightarrow$ B is self-adjoint.

SOLUTION: The proof is almost identical to part b.

4. (10 pts.) Below you must provide a field $\mathbb{F}$ and $A \in \mathrm{M}_{2\times 2}(\mathbb{F})$ such that $L_A\colon \mathbb{F}^2 \to \mathbb{F}^2$ satisfies the stated properties. Your A must be explicit. Make sure you specify what $\mathbb{F}$ is!
a. (5 pts.) L_A is normal but not diagonalizable.

SOLUTION:We must pick $\mathbb{F} = \mathbb{R}$ here. Just about any rotation matrix will do the trick, as such matrices are unitary and have no eigenvalues, as long as the angle is not 0 or π. Let's settle with $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

b. (5 pts.) L_A is a projection but not self-adjoint.

SOLUTION: Any nonorthogonal projection of $\mathbb{R}^2$ will do here. Take the projection on span$\{(1,0)\}$ along span$\{(1,1)\}$, for example. Its matrix is $A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$.

5. (15 pts.) Below information is provided for various $T_i\colon \mathbb{F}^n \to \mathbb{F}^n$. Copy the following table in your bluebook and fill each of the squares with a 'YES' or 'NO'. The inner product on $\mathbb{F}^n$ is always the standard one. Please observe the choice of $\mathbb{F}$ in each case.

	Invertible	Diagonalizable	Normal	Self-Adjoint	Unitary
T_1					
T_2					
T_3					

$T_1\colon \mathbb{R}^3 \to \mathbb{R}^3$ has eigenvalues 0 and 2, where $E_0 = \{(r,s,0)\colon r,s \in \mathbb{R}\}$ and $E_2 = \text{span}\{(1,1,1)\}$.

$T_2\colon \mathbb{C}^3 \to \mathbb{C}^3$ has eigenvalues $i, -i, \frac{\sqrt{2}}{2}(1+i)$, $E_i = \text{span}\{(1,-1,0)\}$, $E_{-i} = \{(1,0,-1)\}$, $E_{\frac{\sqrt{2}}{2}(1+i)} = \text{span}\{(1,1,1)\}$.
$T_3\colon \mathbb{C}^3 \to \mathbb{C}^3$ has eigenvalues 1 and -1, $E_1 = \text{span}\{(1,1,-2),(1,-1,0)\}$ and $E_{-1} = \text{span}\{(1,1,1)\}$.

SOLUTION: The general reasoning here is as follows. The transformation is invertible iff 0 is not an eigenvalue. The transformation is diagonalizable iff $n = \sum \dim E_{\lambda_i}$. When $\mathbb{F} = \mathbb{C}$, we have T normal iff T is diagonalizable and the eigenspaces are mutually orthogonal. Similarly, when $\mathbb{F} = \mathbb{C}$ we have T self-adjoint (resp. unitary) iff T is normal and its eigenvalues are real (resp. of absolute value 1). When $\mathbb{F} = \mathbb{R}$, one must look a little more carefully, as normal does not imply diagonalizable in this case.

	Invertible	Diagonalizable	Normal	Self-Adjoint	Unitary
T_1	NO	YES	NO	NO	NO
T_2	YES	YES	NO	NO	NO
T_3	YES	YES	YES	YES	YES

NOTE: The eigenspaces of T_2 are in fact NOT mutually orthogonal. This was trickier than I meant it to be, and many people took them to be mutually orthogonal...MYSELF INCLUDED! For this reason, I also accepted

	Invertible	Diagonalizable	Normal	Self-Adjoint	Unitary
T_2	YES	YES	YES	NO	YES

as a correct answer for T_2.