Math 53 Spring 2004
Solutions to Midterm 1

1.

$$(\langle 0,4,3\rangle - \langle 1,2,3\rangle) \times (\langle 2,2,1\rangle - \langle 1,2,3\rangle) \cdot (\langle x,y,z\rangle - \langle 1,2,3\rangle) = 0$$

2. (a)

$$\nabla f(x,y,z) = \langle y\cos(xy) + 2zxe^{x^2z}, x\cos(xy), x^2e^{x^2z}\rangle$$

(b)

$$g_{xy}(x,y) = \frac{\partial}{\partial x} g_y(x,y) = \frac{\partial}{\partial x} y\cos(xy) = \cos(xy) - xy\sin(xy)$$

3.

$$\begin{aligned}
\text{Area} &= \int_0^{2\pi} \frac{1}{2}(1-\cos\theta)^2 \, d\theta \\
&= \int_0^{2\pi} \frac{1}{2} - \cos\theta + \frac{\cos^2\theta}{2} \, d\theta \\
&= \pi + \int_0^{2\pi} \frac{\cos^2\theta}{2} \, d\theta \\
&= \pi + \int_0^{2\pi} \frac{1+\cos(2\theta)}{4} \, d\theta \\
&= \frac{3\pi}{2} + \int_0^{2\pi} \frac{\cos(2\theta)}{4} \, d\theta \\
&= \frac{3\pi}{2}
\end{aligned}$$

4. If $f(x,y) = x^y$, then

$$\begin{aligned}
f_x(x,y) &= yx^{y-1} \\
f_y(x,y) &= (\ln x)x^y
\end{aligned}$$

and the linearization of f at $(x,y) = (3,2)$ is

$$\begin{aligned}
L(x,y) &= f_x(3,2)(x-3) + f_y(3,2)(y-2) + f(3,2) \\
&\approx 6(x-3) + 9.9(y-2) + 9
\end{aligned}$$

so the appoximation we're looking for is

$$L(2.99, 2.01) \approx 6(-0.01) + 9.9(0.01) + 9 = 9.039$$

5. If $F(x, y, z) = xyz + e^{\sin z} - x$, then

$$\begin{aligned} F_x(x, y, z) &= yz - 1 \\ F_y(x, y, z) &= xz \\ F_z(x, y, z) &= xy + (\cos z)e^{\sin z} \end{aligned}$$

and therefore

$$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = -\frac{yz - 1}{xy + (\cos z)e^{\sin z}} \\ \frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = -\frac{xz}{xy + (\cos z)e^{\sin z}} \end{aligned}$$

6.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial z}{\partial x} + t\frac{\partial z}{\partial y}$$

and

$$\begin{aligned} \frac{\partial^2 z}{\partial s \partial t} &= \frac{\partial^2 z}{\partial t \partial s} \\ &= \frac{\partial}{\partial t}\frac{\partial z}{\partial s} \\ &= \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x} + t\frac{\partial z}{\partial y}\right) \\ &= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial y \partial x}\frac{\partial y}{\partial t} + t\frac{\partial^2 z}{\partial x \partial y}\frac{\partial x}{\partial t} + t\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y} \\ &= \frac{\partial^2 z}{\partial x^2} + s\frac{\partial^2 z}{\partial y \partial x} + t\frac{\partial^2 z}{\partial x \partial y} + st\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y} \end{aligned}$$

7. (a) If $f(x, y, z) = x^2 + y^2 + z^2 - 1$, then S is defined by the equation $f(x, y, z) = 0$. So, if (a, b, c) is on S, an equation for the tanget is

$$\nabla f(a, b, c) \cdot (\langle x, y, z \rangle - \langle a, b, c \rangle) = 0$$

Expanding this equation yields

$$2a(x-a) + 2b(x-b) + 2c(x-c) = 0$$

which simplifies to

$$ax + by + cz = 1$$

(b) If the tangent to S at (a, b, c) contains $(-1, 2, 1)$ and $(3, -1, 0)$ then we know from part (a) that the following two equations must hold

$$a(-1) + b(2) + c(1) = 1$$
$$a(3) + b(-1) + c(0) = 1$$

These can be used to get b and c in terms of a:

$$b = 3a - 1$$
$$c = 3 - 5a$$

Since (a, b, c) is on S,

$$a^2 + b^2 + c^2 - 1 = 0$$

and thus

$$a^2 + (3a-1)^2 + (3-5a)^2 - 1 = (5a-3)(7a-3) = 0$$

So we have two possible values for (a, b, c):

$$(\frac{3}{5}, \frac{4}{5}, 0) \text{ and } (\frac{3}{7}, \frac{2}{7}, \frac{6}{7})$$