MATH 185 — FINAL EXAM

Problem #1. Write down all the values of

$$(1+i)^{\pi}.$$

Problem #2. Suppose that $f = u(x,y) + iv(x,y)$ is an analytic mapping from a domain D_z in the z-plane to a region D_w in the w-plane. Suppose that $h = h(u,v)$ is harmonic in D_w and $g = g(u,v)$ is a harmonic conjugate of h.

Explain why

$$H(x,y) = h(u(x,y), v(x,y)) \quad \text{and} \quad G(x,y) = g(u(x,y), v(x,y))$$

are harmonic in D_z.

Problem #3. Recall that *Liouville's Theorem* asserts that if f is a bounded analytic function defined on the entire complex plane $\mathbb{C}$, then f is constant.

Prove Liouville's Theorem, starting with the integral formula

$$f'(z_0) = \frac{1}{2\pi i}\int_{C_R} \frac{f(z)}{(z-z_0)^2}dz,$$

where C_R is the positively oriented circle of radius R centered at z_0.

Problem #4. (i) Find a linear fractional transformation T so that

$$T(0) = 1, T(i) = i, T(\infty) = -1.$$

(ii) What is the image of the y-axis under this transformation?

Problem #5. Prove that if

$$f(z) = \frac{\phi(z)}{(z-z_0)^m},$$

where m is a positive integer, ϕ is analytic at z_0 and $\phi(z_0) \neq 0$, then f has a pole of order m at z_0 and

$$\operatorname*{Res}_{z=z_0} f = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

Problem #6. Compute

$$\int_C \frac{1}{z^{10} + 12z^3 + 1}\,dz,$$

where C is a positively oriented circle that surrounds all the roots of $z^{10} + 12z^3 + 1$.

Problem #7. Find the Laurent series for

$$f(z) = \frac{1}{(z-1)^2(z-3)}$$

in the two regions (i) $0 < |z-1| < 2$ and (ii) $0 < |z-3| < 2$.

Problem #8. Compute

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx.$$

Problem #9. Suppose that f has a *simple pole* at the real point x_0.

Prove that

$$\lim_{\rho \to 0} \int_{C_\rho} f\,dz = -i\pi \operatorname*{Res}_{z=x_0} f,$$

where C_ρ denotes the half-circle with center x_0 and radius $\rho > 0$ in the upper half-plane, with clockwise orientation.

Problem #10. Find the principal value of

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 - 2x + 2)}\,dx.$$

Problem #11. Prove that if f is analytic at z_0 and f is not identically equal to 0, then for some $\epsilon > 0$,

$$f(z) \neq 0 \qquad \text{for } 0 < |z - z_0| < \epsilon. \qquad f(z_0) = 0$$

This says that the zeros of an analytic function are isolated (unless the function is identically equal to 0.)