MATH 113 - S2
MID-TERM 2

1 (4 pts) Compute the order of $(3, 4)$ in the group $\mathbf{Z}_9 \times \mathbf{Z}_{30}$.

2 Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 1 & 6 & 8 & 3 & 7 & 5 \end{pmatrix}$$

a (3 pts) Write σ as a product of disjoint cycles.

b (3 pts) Hence write sigma as a product of transpositions, and determine whether σ is an even permutation.

3 (5 pts) Let

$$\phi : G \to G'$$

be a **surjective homomorphism.** Assume G is abelian. Show that G' is also abelian.

4 (5 pts) Let

$$\phi : G \to G'$$

be a **surjective homomorphism** . Assume that G is finite. Show that

$$|G| = |G'| \times |\ker \phi|$$

(Hint: use the fundamental homomorphism theorem.)

MIDTERM 2

SOLUTIONS

4 pts 1) order of 3 in $\mathbb{Z}_9 = \frac{9}{\gcd(3,9)} = \frac{9}{3} = 3$

order of 4 in $\mathbb{Z}_{30} = \frac{30}{\gcd(4,30)} = \frac{30}{2} = 15$

order of $(3,4)$ in $\mathbb{Z}_9 \times \mathbb{Z}_{30} =$
$= \mathrm{lcm}(3, 15) = 15$

3 pts 2 a) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 1 & 6 & 8 & 3 & 7 & 5 \end{pmatrix}$

$\sigma = (1\ 2\ 4\ 6\ 3)(5\ 8)$

3 pts 2 b) $\sigma = (1\,3)(1\,6)(1\,4)(1\,2)(5\,8)$

σ can be written as a product of 5 transpositions

$\Rightarrow \sigma$ is an odd permutation

5 pts 3) Let $a', b' \in G'$

We have to show that $a'b' = b'a'$.

Now ϕ surjective implies that there exist $a, b \in G$ such that $a' = \phi(a)$ and $b' = \phi(b)$

ϕ is a homomorphism and therefore

$a'b' = \phi(a)\phi(b) = \phi(ab)$

But G is abelian, so that

$ab = ba$

and therefore $\phi(ab) = \phi(ba)$

$\phi(ba) = \phi(b)\phi(a) = b'a'$

$\Rightarrow a'b' = b'a'$ $\square$

5 pts 4) Using the hint, we have from the fundamental homomorphism theorem that if G, G' are groups, $\phi: G \to G'$ homomorphism with $H = \text{Kernel } \phi$, then $\phi[G]$ is a group, and $\mu: \frac{G}{H} \to \phi[G]$

$\mu(gH) = \phi(g)$ is an isomorphism.

Now, ϕ surjective means that

$G' = \phi[G]$

and, therefore, in this case we have

$\mu : \frac{G}{H} \rightarrow G'$ isomorphism

This implies that

$$\left| \frac{G}{H} \right| = |G'|$$

But from Lagrange theorem for a finite group G this is given by

$$\left| \frac{G}{H} \right| = \frac{|G|}{|H|}$$

which is the number of left cosets of H, that coincides also with the number of right cosets.

It follows that

$$\frac{|G|}{|H|} = |G'| \Rightarrow |G| = |H||G'| \quad \square$$