UCB Math 128A, Spring 2009: Midterm 1, Solutions

February 20, 2009

Name: ______________________________

SID: ______________________________

GSI: ______________________________

- No books, no notes, no calculators
- Justify all answers
- Do all of the 4 problems
- Exam time 50 minutes

Grading

1	/ 25
2	/ 25
3	/ 25
4	/ 25
	/100

1. (25 points)

Use any method to find an interpolating polynomial $P(x)$ such that

$$P(0) = 0, \quad P(1) = 2, \quad P(2) = 2, \quad P(3) = 0$$

Solution: The Lagrange interpolating polynomial is

$$\begin{aligned}
P(x) &= 2 \cdot \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} + 2 \cdot \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} \\
&= x(x-2)(x-3) - x(x-1)(x-3) \\
&= x^3 - 5x^2 + 6x - (x^3 - 4x^2 + 3x) \\
&= -x^2 + 3x
\end{aligned}$$

2. (25 points)

(a) Show that the fixed point iteration

$$p_n = \frac{p_{n-1}^2 + 3}{5}, \qquad n = 1, 2, \ldots$$

converges for any initial $p_0 \in [0, 1]$.

(b) Estimate how many iterations n are required to obtain an absolute error $|p_n - p|$ less than 10^{-4} when $p_0 = 1$. No numerical value needed, just give an expression for n.

Solution: (a) Show $g(x) \in [0, 1]$ for $x \in [0, 1]$:

$$g(0) = 3/5$$

$$g(1) = 4/5$$

increasing function

Show $|g'(x)| \leq k < 1$:

$$g'(x) = \frac{2x}{5} \leq \frac{2}{5} = k < 1$$

(b)

$$10^{-4} = |p_n - p| \leq k^n \max\{1, 0\} = \left(\frac{2}{5}\right)^n \Rightarrow n \approx \frac{-4}{\log_{10} \frac{2}{5}}$$

3. (25 points)

(a) The following MATLAB code implements the Bisection method. However, there is a fundamental bug in the code. Find and correct the bug.

```
function p=bisection(f,a,b,tol)


while 1
    p=(a+b)/2;
    if p-a<tol, break; end
    if f(b)*f(p)>0
        a=p;
    else
        b=p;
    end
end
```

(b) Give an upper bound for the error $|p_n - p|$ after n steps of the Bisection method on the interval $[a, b]$ (the corrected one, of course).

Solution: (a) `f(a)*f(p)>0` **or** `f(b)*f(p)<0` **or** switch `a=p;` and `b=p;`

(b)

$$|p_n - p| \leq \frac{b-a}{2^n}$$

4. (25 points)

(a) Use Horner's method to evaluate $P(1)$ where

$$P(x) = 2x^3 - 5x^2 + 2x - 4$$

(b) Use your calculations in (a) to find $Q(x)$ and b_0 such that $P(x)$ can be written

$$P(x) = (x-1)Q(x) + b_0$$

Solution: (a) We have $a_3 = 2$, $a_2 = -5$, $a_1 = 2$, $a_0 = -4$. Horner's method gives

$$\begin{aligned} b_3 &= a_3 = 2 \\ b_2 &= b_3 \cdot 1 + a_2 = 2 \cdot 1 - 5 = -3 \\ b_1 &= b_2 \cdot 1 + a_1 = (-3) \cdot 1 + 2 = -1 \\ P(1) = b_0 &= b_1 \cdot 1 + a_0 = (-1) \cdot 1 - 4 = -5 \end{aligned}$$

(b)

$$\begin{aligned} Q(x) &= b_3 x^3 + b_2 x + b_1 = 2x^2 - 3x - 1 \\ b_0 &= -5 \end{aligned}$$