Math 55 Final Exam 11 May 2004

NAME (1 pt): ______________________________

TA (1 pt): ______________________________

Name of Neighbor to your left (1 pt): ______________________________

Name of Neighbor to your right (1 pt): ______________________________

Instructions: This is a closed book, closed notes, closed calculator, closed computer, closed network, open brain exam.

You get one point each for filling in the 4 lines at the top of this page. All other questions are worth 10 points.

Write all your answers on this exam. If you need scratch paper, ask for it, write your name on each sheet, and attach it when you turn it in (we have a stapler).

For full credit justify your answers.

1	
2	
3	
4	
5	
Total	

1

Question 1. (10 points)

Question 1a. (5 points) If 61 people are sitting in a row of 80 chairs, prove that there are at least 4 consecutive occupied chairs.

Answer: *Let the pigeons be the 61 people, and the holes be the 20 disjoint sets of 4 consecutive chairs 1-4, 5-8, 9-12, ... , 77-80. By the Generalized Pigeonhole Principle some hole must get at least* $\lceil 61/20 \rceil = 4$ *pigeons.*

Question 1b. (5 points) Label each of the following sets as "finite", "countably infinite", or "uncountable". Justify your answers (you may cite theorems proven in class).

1. Set of all correct computer programs in Java.
2. Set of all computer programs in Java that have ever been written.
3. Set of rational numbers between 1 and 2.
4. Set of functions with domain $\{0, 1, 2\}$ and codomain $\mathbf{N} = \{1, 2, 3, ...\}$.
5. Set of functions with domain $\mathbf{N}$ and codomain $\{0, 1, 2\}$.

Answer:

1. *Countably infinite, because it is an infinite subset of the countable set consisting of all finite strings of characters on the keyboard.*
2. *Finite, because each of finitely many Java programmers (or Java generating programs) can only have written a finite amount of code in the finite amount of time they have existed, working at a finite speed.*
3. *Countably infinite. The set of all pairs of integers is countably infinite because it is the Cartesian product of two countably infinite sets, by a result in class. Then, the rationals between 1 and 2 are in a one-to-one correspondence with an infinite subset of all pairs* (i, j) *of integers, namely those where* $j \neq 0$, $gcd(i, j) = 1$, *and* $j \leq i \leq 2j$. *Finally, an infinite subset of a countable set is countably infinite by a result shown in class.*
4. *Countably infinite, because the set of functions is in one-to-one correspondence with* $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$.
5. *Uncountable, because we showed in class that the set of all sequences of 0s and 1s is uncountable by a diagonalization argument, and the set of all sequences of 0s, 1s and 2s is only larger.*

Question 1. (10 points)

Question 1a. (5 points) If 81 cars are parked in a row of 100 parking places, prove that there are at least 5 consecutive occupied parking places.

Answer: *Let the pigeons be the 81 cars, and the holes be the 20 disjoint sets of 5 consecutive parking places 1-5, 6-10, 11-15, ... , 96-100. By the Generalized Pigeonhole Principle some hole must get at least* $\lceil 81/20 \rceil = 5$ *pigeons.*

Question 1b. (5 points) Label each of the following sets as "uncountable", "countably infinite", or "finite". Justify your answers (you may cite theorems proven in class).

1. Set of rational numbers between .5 and 1.
2. Set of all correct computer programs in C.
3. Set of functions with domain $\{-1, 0, +1\}$ and codomain $\mathbf{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.
4. Set of functions with domain $\mathbf{Z}$ and codomain $\{-1, 0, +1\}$.
5. Set of all computer programs in C that have ever been written.

Answer:

1. *Countably infinite. The set of pairs of integers is countably infinite, because it is the Cartesian product of two countably infinite sets, by a result in class. Then, the rationals between .5 and 1 are in a one-to-one correspondence with an infinite subset of all pairs* (i, j) *of integers, namely those where* $j \neq 0$, $gcd(i, j) = 1$, *and* $j/2 \leq i \leq j$. *Finally, an infinite subset of a countable set is countably infinite by a result shown in class.*
2. *Countably infinite, because it is an infinite subset of the countable set consisting of all finite strings of characters on the keyboard.*
3. *Countably infinite, because the set of functions is in one-to-one correspondence with* $\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}$, *and the Cartesian product of 3 countably infinite sets is countably infinite.*
4. *Uncountable, because we showed in class that the set of all sequences of 0s and 1s is uncountable by a diagonalization argument, and the set of all sequences of* -1*s, 0s and 1s is only larger.*
5. *Finite, because each of finitely many C programmers (or C generating programs) can only have written a finite amount of code in the finite amount of time they have existed, working at a finite speed.*

Question 2 (10 points)

Question 2a (5 points) Prove algebraically that $C(n,r) \cdot C(n-r,k) = C(n,k) \cdot C(n-k,r)$.
Answer:

$$\begin{aligned} C(n,r) \cdot C(n-r,k) &= \frac{n!}{r!(n-r)!} \cdot \frac{(n-r)!}{k!(n-r-k)!} \\ &= \frac{n!}{r!} \cdot \frac{1}{k!(n-r-k)!} \\ &= \frac{n!}{k!} \cdot \frac{1}{r!(n-r-k)!} \\ &= \frac{n!}{k!(n-k)!} \cdot \frac{(n-k)!}{r!(n-r-k)!} \\ &= C(n,k) \cdot C(n-k,r) \end{aligned}$$

Question 2b. (5 points) Prove the same identity using a counting argument.

Answer: *We count the number of ways to choose a subset of r elements and a disjoint subset of k elements from a set of n elements. (1) First choose r elements (there are $C(n,r)$ ways) and then choose k elements from the remaining $n-r$ elements (there are $C(n-r,k)$ ways). Since each pair of choices leads to a different pair of subsets, the product is the number of ways, namely $C(n,r) \cdot C(n-r,k)$. (2) First choose k elements and then r elements from the remaining $n-k$. The same argument yields the expression $C(n,k) \cdot C(n-k,r)$.*

Question 2 (10 points)

Question 2a (5 points) Prove algebraically that $C(m,s)\cdot C(m-s,j) = C(m,j)\cdot C(m-j,s)$.
Answer:

$$\begin{aligned}
C(m,s)\cdot C(m-s,j) &= \frac{m!}{s!(m-s)!}\cdot\frac{(m-s)!}{j!(m-s-j)!} \\
&= \frac{m!}{s!}\cdot\frac{1}{j!(m-s-j)!} \\
&= \frac{m!}{j!}\cdot\frac{1}{s!(m-s-j)!} \\
&= \frac{m!}{j!(m-j)!}\cdot\frac{(m-j)!}{s!(m-s-j)!} \\
&= C(m,j)\cdot C(m-j,s)
\end{aligned}$$

Question 2b. (5 points) Prove the same identity using a counting argument.

Answer: *We count the number of ways to choose a subset of s elements and a disjoint subset of j elements from a set of m elements. (1) First choose s elements (there are $C(m,s)$ ways) and then choose j elements from the remaining $m-s$ elements (there are $C(m-s,j)$ ways). Since each pair of choices leads to a different pair of subsets, the product is the number of ways, namely $C(m,s)\cdot C(m-s,j)$. (2) First choose j elements and then s elements from the remaining $m-j$. The same argument yields the expression $C(m,j)\cdot C(m-j,s)$.*

Question 3. (10 points)

Question 3a. (5 points) For what integer values of n does $141s + 264t = n$ have integer solutions s and t?

Answer: *Either by using the Euclidean algorithm or prime factorization,* $gcd(141, 264) = 3$*. Then we know by the extended Euclidean algorithm that there are integers* s' *and* t' *such that* $141s' + 264t' = 3$*, namely* $s' = 15$ *and* $t' = -8$*. So if* n *is any multiple of* 3 *then a solution* $s = s'n/3$ *and* $t = t'n/3$ *exists, and since* $3|141$ *and* $3|264$*, we get* $3|n$*, so a solution exists only if* $3|n$ *as well.*

Question 3b (5 points) Find the set of all solutions to $141x \equiv 33 \bmod 264$.

Answer: *The extended Euclidean Algorithm shows that* $3 = 15 \cdot 141 - 8 \cdot 264$ *or* $33 = 11 \cdot 3 = 11 \cdot 15 \cdot 141 - 11 \cdot 8 \cdot 264 = 165 \cdot 141 - 88 \cdot 264$*, so* $x' = 165$ *is one solution. Two different solutions* x' *and* x *satisfy* $141(x' - x) \equiv 0 \bmod 264$*, or* $264|141(x' - x)$ *or* $88|47(x' - x)$ *or* $88|(x' - x)$ *since* 88 *and* 47 *are relatively prime or* $x = x' + m \cdot 88 = 165 + m \cdot 88$ *for any* m *or* $x \equiv 165 \bmod 88 \equiv 77 \bmod 88$ *is the general solution.*

Question 3. (10 points)

Question 3a. (5 points) For what integer values of r does $261n + 147m = r$ have integer solutions n and m?

Answer: *Either by using the Euclidean algorithm or prime factorization, $gcd(261, 147) = 3$. Then we know by the extended Euclidean algorithm that there are integers n' and m' such that $261n' + 147m' = 3$, namely $n' = -9$ and $m' = 16$. So if r is any multiple of 3 then a solution $n = n'r/3$ and $m = m'r/3$ exists, and since $3|261$ and $3|147$, we get $3|r$, so a solution exists only if $3|r$ as well.*

Question 3b (5 points) Find the set of all solutions to $261y \equiv 42 \bmod 147$.

Answer: *The extended Euclidean Algorithm shows that $3 = -9 \cdot 261 + 16 \cdot 147$ or $42 = 14 \cdot 3 = 14 \cdot (-9) \cdot 261 + 14 \cdot 16 \cdot 147 = -126 \cdot 261 + 224 \cdot 147$, so $y' = -126$ is one solution. Two different solutions y' and y satisfy $261(y' - y) \equiv 0 \bmod 147$, or $147|261(y' - y)$ or $49|87(y'-y)$ or $49|(y'-y)$ since 49 and 87 are relatively prime or $y = y'+m\cdot 49 = -126+m\cdot 49$ for any m or $y \equiv -126 \bmod 49 \equiv 21 \bmod 49$ is the general solution.*

Question 4. (10 points) Rosencrantz has an unfair coin that comes up heads 2/3 of the time and tails 1/3 of the time. He plays the following game: first he flips the coin repeatedly until it has come up tails a total of twice. If the first two tails are consecutive, then he scores zero points. If they are non-consecutive, then he scores a number of points equal to the number of flips up to and including the first tail. Let f be the random variable which is the number of points he scores.

Question 4a. (5 points) Find the generating function $G(x)$ for the random variable f.

Answer: *Note that* $P(f=0)=\frac{1}{3}$, *since the chance of getting another tail right after the first tail (or at any flip, for that matter) is* $\frac{1}{3}$. *For* $k>0$, $P(f=k)=(\frac{2}{3})^{k-1}\cdot\frac{1}{3}\cdot\frac{2}{3}$, *since winning* k *points means that Rosencrantz gets* $k-1$ *heads in a row, then one tail, then one head again (further flips do not affect the outcome). Thus*

$$\begin{aligned} G(x) &= \frac{1}{3}+\sum_{k=1}^{\infty} x^k\cdot(\frac{2}{3})^{k-1}\cdot\frac{1}{3}\cdot\frac{2}{3} \\ &= \sum_{k=0}^{\infty}\frac{1}{3}\cdot(\frac{2x}{3})^k \\ &= \frac{1}{3}\cdot\frac{1}{1-\frac{2x}{3}} \\ &= \frac{1}{3-2x} \end{aligned}$$

Rosencrantz and Guildenstern were minor characters in Shakespeare's Hamlet. For their connection to coin tossing, see the first scene of Stoppard's play "Rosencrantz and Guildenstern Are Dead."

Question 4b. (5 points) Compute $E(f)$ and $V(f)$.

Answer: $G'(x)=\frac{2}{(3-2x)^2}$ *so* $E(f)=G'(1)=2$. $G''(x)=\frac{8}{(3-2x)^3}$ *so* $V(f)=G''(1)+G'(1)-(G'(1))^2=8+2-2^2=6$.

Question 4. (10 points) Guildenstern has an unfair coin that comes up heads 1/4 of the time and tails 3/4 of the time. He plays the following game: first he flips the coin repeatedly until it has come up heads a total of twice. If the first two heads are consecutive, then he scores zero points. If they are non-consecutive, then he scores a number of points equal to the number of flips up to and including the first head. Let f be the random variable which is the number of points he scores.

Question 4a. (5 points) Find the generating function $G(x)$ for the random variable f.

Answer: *Note that* $P(f=0)=\frac{1}{4}$, *since the chance of getting another head right after the first head (or at any flip, for that matter) is* $\frac{1}{4}$. *For* $k>0$, $P(f=k)=(\frac{3}{4})^{k-1}\cdot\frac{1}{4}\cdot\frac{3}{4}$, *since winning* k *points means that Guildenstern gets* $k-1$ *tails in a row, then one head, then one tail again (further flips do not affect the outcome). Thus*

$$\begin{aligned} G(x) &= \frac{1}{4}+\sum_{k=1}^{\infty} x^k\cdot(\frac{3}{4})^{k-1}\cdot\frac{1}{4}\cdot\frac{3}{4} \\ &= \sum_{k=0}^{\infty}\frac{1}{4}\cdot(\frac{3x}{4})^k \\ &= \frac{1}{4}\cdot\frac{1}{1-\frac{3x}{4}} \\ &= \frac{1}{4-3x} \end{aligned}$$

Rosencrantz and Guildenstern were minor characters in Shakespeare's Hamlet. For their connection to coin tossing, see the first scene of Stoppard's play "Rosencrantz and Guildenstern Are Dead."

Question 4b. (5 points) Compute $E(f)$ and $V(f)$.

Answer: $G'(x)=\frac{3}{(4-3x)^2}$ *so* $E(f)=G'(1)=3$. $G''(x)=\frac{18}{(4-3x)^3}$ *so* $V(f)=G''(1)+G'(1)-(G'(1))^2=18+3-3^2=12$.

Question 5 (10 points) Sort the following functions into increasing order using $O(\cdot)$ notation. For example, one possible answer is $f_1 = O(f_2)$, $f_2 = O(f_3)$, ... Justify your answers.

- $f_1(n) = \sum_{j=1}^{n} j^4$
- $f_2(n) = n!$ Hint: Stirling's formula says $n! \approx \sqrt{2\pi} n^{n+1/2} e^{-n}$.
- $f_3(n) = \log(\log(n^{n^n}))$
- $f_4(n) = 6^{2^n}/2^{6^n}$
- $f_5(n) = \sum_{j=1}^{n} 4^j$

Answer: *Some simplifications and approximations:*

- $f_1(n) = n^5/5 + O(n^4)$
- $f_2(n) \approx \sqrt{2\pi}\sqrt{n} \cdot e^{-n} n^n$ *by Stirling's formula*
- $f_3(n) = \log(n^n \log n) = n \log n + \log \log n$
- $\log f_4(n) = 2^n \log 6 - 6^n \log 2 \to -\infty$ *as* $n \to \infty$ *so* $f_4(n) \to 0$.
- $f_5(n) = (4^{n+1} - 1)/3$

Thus $f_4(n) = O(f_3(n))$ *since* $f_1(n) \to 0$ *and* $f_3(n) \to \infty$.
Then $f_3(n) = O(f_1(n))$ *since* $f_1(n)$ *grows like* n^5 *and* $f_3(n)$ *grows like* $n \log n$.
Then $f_1(n) = O(f_5(n))$ *since* $f_5(n)$ *grows exponentially (like* 4^n*) and* $f_1(n)$ *just grows like a polynomial (*n^5*).*
Then $f_5(n) = O(f_2(n))$ *since* $\log f_5(n) = O(n)$ *and* $\log f_2(n) \approx (n+1/2)\log n - n + \log\sqrt{2\pi}$ *grows like* $n \log n$.

Question 5 (10 points) Sort the following functions into increasing order using $O(\cdot)$ notation. For example, one possible answer is $g_5 = O(g_1)$, $g_1 = O(g_3)$, ... Justify your answers.

- $g_1(m) = m!$ Hint: Stirling's formula says $m! \approx \sqrt{2\pi} m^{m+1/2} e^{-m}$.
- $g_2(m) = 7^{3^m}/3^{7^m}$
- $g_3(m) = \log(\log(m^{m^{2m}}))$
- $g_4(m) = \sum_{k=1}^{m} k^5$
- $g_5(m) = \sum_{k=1}^{m} 7^k$

Answer: *Some simplifications and approximations:*

- $g_1(m) \approx \sqrt{2\pi}\sqrt{m} \cdot e^{-m} m^m$ *by Stirling's formula*
- $\log g_2(m) = 3^m \log 7 - 7^m \log 3 \to -\infty$ *as* $m \to \infty$ *so* $g_2(m) \to 0$.
- $g_3(m) = \log(m^{2m} \log m) = 2m \log m + \log\log m$
- $g_4(m) = m^6/6 + O(m^5)$
- $g_5(m) = (7^{m+1} - 1)/6$

Thus $g_2(m) = O(g_3(m))$ *since* $g_2(m) \to 0$ *and* $g_3(m) \to \infty$.

Then $g_3(m) = O(g_4(m))$ *since* $g_4(m)$ *grows like* m^6 *and* $g_3(m)$ *grows like* $m \log m$.

Then $g_4(m) = O(g_5(m))$ *since* $g_5(m)$ *grows exponentially (like* 7^m*) and* $g_4(m)$ *just grows like a polynomial (*m^6*).*

Then $g_5(m) = O(g_1(m))$ *since* $\log g_5(m) = O(m)$ *and* $\log g_1(m) \approx (m + 1/2)\log m - m + \log\sqrt{2\pi}$ *grows like* $m \log m$.