Mathematics 1B. Fall Semester 2006 **Professor: Daniel Tataru**

Midterm 1, Solutions

(20) 1. Evaluate the following (indefinite) integrals

a) $\int e^{\sqrt{x}} dx$

Solution: Substitute $x = u^2$, $dx = 2udu$. The integral becomes

$$\int 2ue^u du$$

We integrate by parts to obtain

$$\int 2ue^u du = 2ue^u - \int 2e^u du = 2ue^u - 2e^u + C = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

b) $\int x \tan^2 x \, dx$

Solution: We rewrite the integral as

$$\int x \tan^2 x \; dx = \int x(\sec^2 x - 1) \; dx = \int x \sec^2 x \; dx - \frac{x^2}{2}$$

The first term is integrated by parts,

$$\int x \sec^2 x \; dx = x \tan x - \int \tan x dx = x \tan x + \ln|\cos x| + C$$

The final result is

$$\int x \tan^2 x \; dx = x \tan x + \ln|\cos x| - \frac{x^2}{2} + C$$

(20) 2. Evaluate the following (definite) integrals:

a) $\int_{-\infty}^{\infty} \frac{4x^2}{x^4+4}dx$

Solution: We use partial fractions. First we factor the denominator,

$$x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 = (x^2+2)^2 - (2x)^2 = (x^2+2x+2)(x^2-2x+2)$$

Then we decompose into partial fractions,

$$\frac{4x^2}{x^4+4} = \frac{Ax+B}{x^2-2x+2} - \frac{Cx+D}{x^2+2x+2}$$

This gives

$$4x^2 = (Ax+B)(x^2+2x+2) + (Cx+D)(x^2-2x+2)$$

Identifying the coefficients we obtain the equations

$$A + C = 0, \quad 2A + B - 2C + D = 4, \quad 2A + 2B + 2C - 2D = 0, \quad 2B + 2D = 0$$

which has solutions $A = 1, B = 0, C = -1, D = 0$. Hence the indefinite integral becomes

$$\int \frac{x}{x^2-2x+2} - \frac{x}{x^2+2x+2}dx = \int \frac{x-1}{(x-1)^2+1} + \frac{1}{(x-1)^2+1} - \frac{x+1}{(x+1)^2+1} + \frac{1}{(x+1)^2+1}dx$$

$$= \frac{1}{2}\ln((x-1)^2+1) + \tan^{-1}(x-1) - \frac{1}{2}\ln((x+1)^2+1) + \tan^{-1}(x+1) + C$$

$$= \frac{1}{2}\ln\frac{(x-1)^2+1}{(x+1)^2+1} + \tan^{-1}(x-1) + \tan^{-1}(x+1) + C$$

To find the definite integral we evaluate this between $-\infty$ and ∞. Since

$$\lim_{x\to\pm\infty} \frac{(x-1)^2+1}{(x+1)^2+1} = 1$$

we are left only with the contributions from the last two terms,

$$\int_{-\infty}^{\infty} \frac{4x^2}{x^4+4}dx = [\tan^{-1}(x-1) + \tan^{-1}(x+1)]|_{-\infty}^{\infty} = 2\pi$$

b) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin^2 x}}dx$

Solution: First substitute $u = \sin x$, $du = \cos x dx$. The integral becomes

$$\int_0^1 \frac{1}{\sqrt{1+u^2}}du$$

Then substitute $u = \tan\theta$, $du = \sec^2\theta d\theta$, transforming the integral into

$$\int_0^{\pi/4} \frac{\sec^2\theta}{\sec\theta}d\theta = \int_0^{\pi/4} \sec\theta d\theta = \ln|\sec\theta + \tan\theta||_0^{\pi/4} = \ln(\sqrt{2}+1)$$

(20) 3. a) Suppose that $f(x)$ is a function defined on $[a,b]$. State the formula for the area of the surface of revolution obtained by rotating the graph of f around the y axis.

Solution:

$$A = 2\pi \int_a^b x\sqrt{1+f'(x)^2}dx$$

b) Find that area in the case when $f(x) = 3x^{1/3}$ and $a = 0$, $b = 1$.

Solution: We have

$$A = 2\pi \int_0^1 x\sqrt{1+x^{-4/3}}dx$$

Substituting $x = u^3$, $dx = 3u^2du$ we transform this into

$$2\pi \int_0^1 3u^5\sqrt{1+u^{-4}}du = 2\pi \int_0^1 3u^3\sqrt{u^4+1}du$$

Setting $u^4+1 = v$, $4u^3du = dv$ the integral becomes

$$A = \pi \int_1^2 \frac{3}{2}\sqrt{v}dv = \pi v^{\frac{3}{2}}|_1^2 = \pi(2\sqrt{2}-1)$$

(20) 4. Determine (providing an explanation) the convergence or divergence of the following series:

a) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

Solution: Use the integral test to compare with the integral

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}dx$$

Substituting $\ln x = u$, $x^{-1}dx = du$ the indefinite integral turns into

$$\int \frac{1}{\sqrt{u}}du = 2\sqrt{u} + C = 2\sqrt{\ln x} + C$$

Then for the improper integral we get

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}dx = \lim_{b\to\infty} 2\sqrt{\ln b} - 2\sqrt{\ln 2} = \infty$$

Hence the improper integral diverges. Then the series is also divergent.

b) $\sum_{n=1}^{\infty} \frac{1+(-1)^n n}{n^2+2n}$

Solution: We split the series in two,

$$\frac{1+(-1)^n n}{n^2+2n} = \frac{1}{n^2+2n} + \frac{(-1)^n}{n+2}$$

We have $\frac{1}{n^2+2n} \leq \frac{1}{n^2}$ therefore the series $\sum \frac{1}{n^2+2n}$ converges by comparison with the p-series.

On the other hand the series $\sum \frac{(-1)^n}{n+2}$ converges due to the alternating test.

Summing up the two series we conclude that the original series converges.

c) $\sum_{n=1}^{\infty} \frac{(n!)^2}{e^{n^2}}$

Solution: Use ratio test:

$$\lim_{n\to\infty} \frac{\frac{((n+1)!)^2}{e^{(n+1)^2}}}{\frac{(n!)^2}{e^{n^2}}} = \lim_{n\to\infty} \frac{(n+1)^2}{e^{2n+1}}$$

We compute this limit using L'Hopital's rule,

$$\lim_{x\to\infty} \frac{(x+1)^2}{e^{2x+1}} = \lim_{x\to\infty} \frac{x+1}{e^{2x+1}} = \lim_{x\to\infty} \frac{1}{2e^{2x+1}} = 0$$

By the ratio test it follows that the series is convergent.

(20) 5. a) Estimate the error in approximating the following series by the sum of its first 10 terms:

$$\sum_{n=1}^{\infty} \frac{1}{n^4+n^2}$$

Solution: We first estimate $\frac{1}{n^4+n^2} \leq \frac{1}{n^4}$. Since the function x^{-4} is decreasing, the error is estimated in terms of the integral,

$$|R_n| \leq \int_n^\infty \frac{1}{x^4}dx = -\frac{1}{3x^3}|_n^\infty = \frac{1}{3n^3}$$

Hence

$$|R_{100}| \leq \frac{1}{3000000}$$

b) Estimate the partial sums of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

Solution: The series is a p-series which diverges. Since the function x^{-4} is decreasing, we can compare the partial sums with the corresponding integral,

$$S_n \approx \int_1^n \frac{1}{\sqrt{x}}dx = \frac{1}{2}(\sqrt{n}-1)$$

c) Compute the sum of the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

Using partial fractions we write

$$\frac{1}{n^2-1} = \frac{1}{2}(\frac{1}{n-1} - \frac{1}{n+1})$$

Then the series is a telescopic sum. Its partial sums are

$$S_n = \frac{1}{2}(\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1})$$

Almost all terms cancel, and we obtain

$$S_n = \frac{1}{2}(\frac{1}{1} + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}) \to \frac{3}{4}$$

Hence the sum of the series is 3/4.