121A - Spring 09

Midterm exam - March 16

Lecturer: Dr. Shamgar Gurevich.

Instructions. Do only three a's, b's and c's.

1. (Linear Algebra)

   (a) (8) Let V be a vector space. Define when a map $T: V \to V$ is a *linear operator.*

   (b) (15) Consider the vector space $V =$ *The Plane*. Let $R_\theta : V \to V$ be the map of rotation of the plane by angle θ around $(0,0)$. Show that R_θ is a linear operator.

   (c) (10) Consider the standard inner product on V given by the formula $\langle u, v\rangle = \cos\alpha(u,v)\cdot\|u\|\cdot\|v\|$, where $\alpha(u,v)$ is the angle between u and v and $\|u\|$ is the length of a vector u. Show that $\langle,\rangle$ is invariant under R_θ, i.e., show that $\langle R_\theta u, R_\theta v\rangle = \langle u, v\rangle$ for every $u, v \in V$ (Clue: There is nothing to compute!).

2. (Linear Algebra)

   (a) (8) Let V be a vector space. Define when a subset $B \subset V$ is a *basis* for V. Define what is $\dim(V)$.

   (b) (15) Find a basis B for the vector space $V = \{A \in M_2(\mathbb{R});\ tr(A) = 0\}$. Compute $\dim(V) =?$.

   (c) (10) Compute the vector of coordinates

   $$\left[\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\right]_B =?,$$

   where B is the basis you defined in b. above.

3. (Complex numbers)

   (a) (8) Formulate De Moivre's theorem about the series $e^z = \sum \frac{z^n}{n!}$.

   (b) (15) Prove De Moivre's theorem (Clue: $(z+w)^n = \sum \binom{n}{l} z^l w^{n-l}$ and $(\sum \frac{z^n}{n!})\cdot(\sum \frac{w^m}{m!})$). Explain where you use absolutely convergence of the series.

   (c) (10) Compute all the complex solutions of the equation $z^4 = -1$. Draw the solutions in the plane.

4. (Complex functions)

   (a) (8) Formulate Cauchy-Riemann (C-R) theorem about a function $f(x+iy) = u(x,y) + iv(x,y)$ which is analytic in an open subset $U \subset \mathbb{C}$.

   (b) (15) Prove C-R theorem (Clue: The chain rule).

   (c) (10) Let f be an analytic function in an open subset $U \subset \mathbb{C}$.Consider the associated function $u(x,y)$. Show that u solve the Laplace differential equation $\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0$ (Clue: You are free to use the fact that $\frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}$ and of course the C-R equations).

5. (Taylor series)

   (a) (8) Formulate Taylor's theorem about the unique presentation of an analytic function $f: U \subset \mathbb{C} \to \mathbb{C}$ as a power series around a point $z_0 \in U$.

   (b) (15) Let $f(z)$ be an analytic function in an open subset $U \subset \mathbb{C}$. Suppose that in a small open ball $B \subset U$ around z_0 the function f has the Taylor series presentation

   $$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \qquad (1)$$

   Show that $a_n = \frac{f^{(n)}(z_0)}{n!}$. Explain where you use the absolutely convergence of the series (1).

(c) (10) Using the Taylor series around $z_0 = 0$, compute all the derivatives of the function $f(z) = \frac{1}{1+z^2}$. Explain where you use the uniqueness of the Taylor series that represent f around 0 (Clue: The Geometric series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$ for x with $|x| < 1$).

Good Luck!!!