121A - Spring 09 Midterm exam - March 16

Lecturer: Dr. Shamgar Gurevich. Instructions. Do only three a's, b's and c's.

- 1. (Linear Algebra)
 - (a) (8) Let V be a vector space. Define when a map $T: V \to V$ is a linear operator.
 - (b) (15) Consider the vector space V = The Plane. Let $R_{\theta} : V \to V$ be the map of rotation of the plane by angle θ around (0,0). Show that R_{θ} is a linear operator.
 - (c) (10) Consider the standard inner product on V given by the formula $\langle u, v \rangle = \cos \alpha(u, v) \cdot ||u|| \cdot ||v||$, where $\alpha(u, v)$ is the angle between u and v and ||u|| is the length of a vector u. Show that \langle , \rangle is invariant under R_{θ} , i.e., show that $\langle R_{\theta}u, R_{\theta}v \rangle = \langle u, v \rangle$ for every $u, v \in V$ (Clue: There is nothing to compute!).
- 2. (Linear Algebra)
 - (a) (b) Let V be a vector space. Define when a subset $B \subset V$ is a basis for V. Define what is dim(V).
 - (b) (15) Find a basis B for the vector space $V = \{A \in M_2(\mathbb{R}); tr(A) = 0\}$. Compute dim(V) = ?.
 - (c) (10) Compute the vector of coordinates

$$\begin{bmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \end{bmatrix}_B = ?,$$

where B is the basis you defined in b. above.

- 3. (Complex numbers)
 - (a) (8) Formulate De Moivre's theorem about the series $e^z = \sum \frac{z^n}{n!}$.
 - (b) (15) Prove De Moivre's theorem (Clue: $(z+w)^n = \sum {\binom{n}{l}} z^l w^{n-l}$ and $(\sum \frac{z^n}{n!}) \cdot (\sum \frac{w^m}{m!})$). Explain where you use absolutely convergence of the series.
 - (c) (10) Compute all the complex solutions of the equation $z^4 = -1$. Draw the solutions in the plane.
- 4. (Complex functions)
 - (a) (8) Formulate Cauchy-Riemann (C-R) theorem about a function f(x + iy) = u(x, y) + iv(x, y)which is analytic in an open subset $U \subset \mathbb{C}$.
 - (b) (15) Prove C-R theorem (Clue: The chain rule).
 - (c) (10) Let f be an analytic function in an open subset $U \subset \mathbb{C}$.Consider the associated function u(x, y). Show that u solve the Laplace differential equation $\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} = 0$ (Clue: You are free to use the fact that $\frac{\partial}{\partial x} \frac{\partial}{\partial y} = \frac{\partial}{\partial y} \frac{\partial}{\partial x}$ and of course the C-R equations).
- 5. (Taylor series)
 - (a) (8) Formulate Taylor's theorem about the unique presentation of an analytic function $f: U \subset \mathbb{C} \to \mathbb{C}$ as a power series around a point $z_0 \in U$.
 - (b) (15) Let f(z) be an analytic function in an open subset $U \subset \mathbb{C}$. Suppose that in a small open ball $B \subset U$ around z_0 the function f has the Taylor series presentation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$
 (1)

Show that $a_n = \frac{f^{(n)}(z_0)}{n!}$. Explain where you use the absolutely convergence of the series (1).

(c) (10) Using the Taylor series around $z_0 = 0$, compute all the derivatives of the function $f(z) = \frac{1}{1+z^2}$. Explain where you use the uniqueness of the Taylor series that represent f around 0 (Clue: The Geometric series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ for x with |x| < 1).

Good Luck!!!

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(b) ((5) Prove C-E theorem (Close The chain tale).

(c) (1)) Let (be an analytic function in an open subset $U \in \mathbb{C}$. Consider the associated function u(x, y). Show that u solve the Laplace differential equation $\frac{2\pi}{2} + \frac{2\pi}{2} = 0$ (Clue. You are free to use the C-B equation).

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(4)-(3) Faraulate Baylar's theorem about the unique presentation of an analytic function $I:U\subset \mathbb{C}$. $\mathbb{C}\to\mathbb{C}$ as a power series around a point $z_0\in U$.

(b) (b) Let f(z) be an analytic function in an open subset $U \subset \mathbb{C}$. Suppose that in a small open ball $H \subset V'$ stound is the function f has the Taylor series presentation

 $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \qquad (1)$

Show that $a_{i} = \frac{P^{-1}(\omega)}{2}$. Explain where you use the absolutely convergence of the series (1)