Final Solution Set—May 18, 2005

Work each problem on a separate sheet of paper. Be sure to put your name, your section number, and your GSI's name on each sheet of paper. Also, at the top of the page, in the center, write the problem number, and be sure to put the pages in order. Write clearly—explanations (with complete sentences when appropriate) will help us understand what you are doing. Math 49 students taking the linear algebra portion should work problems 1–5; Math 49 students taking the differential equation should work problems 6–10.

1. Let $A := \begin{pmatrix} 1 & -2 & 1 & 1 & -2 \\ -1 & 2 & -1 & 0 & 1 \\ 1 & -2 & 1 & -1 & 0 \end{pmatrix}$.

   (2 points each)

   (a) Find a matrix B which is in reduced row echelon form and which is row equivalent to A.

   $$B = \begin{pmatrix} 1 & -2 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   (b) Find a basis for the null space of A.

   $$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

   (c) Find a basis for the column space of A from among the columns of A.

   $$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

   (d) Find all X such that $AX = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

   $$X = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + X',$$

   where X' is any element of the null space of A.

(e) Find at least one X such that $A^T AX = A^T B$, where $B = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

Hint: you do not need to calculate $A^T A$ to do this.
Recall that the above equation says that AX is the orthogonal projection B' of B on the column space W of A. Luckily we have above an orthogonal basis w_1, w_2 for W, and so

$$\begin{aligned} B' &= \frac{(B|w_1)}{(w_1|w_1)} w_1 + \frac{(B|w_2)}{(w_2|w_2)} w_2 \\ &= 2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \end{aligned}$$

Hence $X = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ plus any element of the null space of A will do.

2. Write the definition of each of the following concepts. Use complete sentences and be as precise as you can.
(2 points each)

(a) The *inverse* of a matrix.
Let A be an $n \times n$ matrix. Then the inverse of A is a matrix B such that $BA = AB = I$; such a matrix is unique if it exists. It can be proved that either of these conditions implies the other, but only for $n \times n$ matrices.

(b) A *linearly independent sequence* in a vector space V.
A sequence $(v_1, v_2, \dots v_n)$ is linearly independent if for every sequence of numbers $(c_1, c_2, \dots c_n)$ such that $c_1 v_1 + \cdots c_n v_n = 0$, each $c_i = 0$.

(c) The *dimension* of a vector space. State the theorem which makes this definition meaningful.
The dimension of V is the number of elements in a basis for V. To know this makes sense, we need to use the theorems that say (1) V has a basis and (2) any two bases have the same number of elements.

(d) The *orthogonal projection* of a vector in $\mathbf{R}^n$ onto a linear subspace W.
The orthogonal projection of v onto W is the vector w in W such that $v - w \in W^\perp$. Equivalently, it is the vector in W which minimizes $||v - w||$.

(e) An *eigenvector* of a linear operator $T: V \to V$.
An eigenvector of V is a vector v such that $Tv = \lambda v$ for some scalar λ.

3. Let V be the space of vectors in $\mathbf{R}^4$ such that $x_1 + x_2 + x_3 + x_4 = 0$. and let W be the set of vectors in V such that $x_1 = x_4$.

   (a) (3 pts). Find an orthogonal basis (v_1, v_2, v_3) for V with $v_1 = (0, 1, -1, 0)$ and such that (v_1, v_2) is a basis for W.
   $v_1 = (0, 1, -1, 0), v_2 = (1, -1, -1, 1), v_3 = (1, 0, 0, -1)$.

   (b) (4 pts). Find the orthogonal projection of $v := (2, 1, 3, -6)$ on W.

$$\begin{aligned} \pi_W(v) &= \frac{(v|v_1)}{(v_1|v_1)} v_1 + \frac{(v|v_2)}{(v_2|v_2)} v_2 \\ &= -v_1 - 2v_2 \\ &= (-2, 1, 3, -2) \end{aligned}$$

   (c) 3 pts). Find the distance from v to W.
   This is

$$||v - \pi_W(v)|| = ||(4, 0, 0, -4)|| = 4\sqrt{2}$$

4. Suppose that A is a matrix with 4 rows and 8 columns, and suppose that the rows of A span a three dimensional subspace of $\mathbf{R}^8$. Answer the following questions, explaining you reasoning.

   (a) (2 pts.) What is the dimension of the space spanned by the columns of A?
   The column space of a matrix has the same dimension as the row space, so the dimension is 3.

   (b) (2 pts.) What is the dimension of the null space of A?
   The rank plus the nullity is $n = 8$, so the answer is $8 - 3 = 5$.

   (c) (2 pts.) What is the dimension of the null space of $A^T A$?
   The null space of $A^T A$ is the same as the null space of A (see below) so the answer is again 5.

   (d) (4 pts.) Prove that for any $m \times n$ real matrix A, the ranks of $A^T A$ and of A are the same
   $A^T A$ is an $n \times n$ matrix and A is an $m \times n$ matrix, so by the rank-nullity formula, it is enough to prove that the dimensions of the null spaces are the same. In fact we prove the null spaces are the same. Evidently $AX = 0$ implies $A^T AX = 0$. Conversely, if $A^T AX = 0$, then $(A^T AX|X) = 0$, hence $(AX|AX) = 0$, hence $||AX|| = 0$, hence $AX = 0$.

5. Let $A = \begin{pmatrix} 6 & -1 \\ 4 & 2 \end{pmatrix}$.

(a) (2 pts.) Find the eigenvalues of A.
These are the roots of the equation $\lambda^2 - 8\lambda + 16$. Thus $\lambda = 4$.

(b) (3 pts.) We know that there exist matrices S and T such that $A = STS^{-1}$, where T is upper triangular. Find T.
T can be $\begin{pmatrix} 4 & a \\ 0 & 4 \end{pmatrix}$, for any nonzero value of a. (An answer with any $a \neq 0$ is acceptable here.)

(c) (3 pts.) Now compute e^{tA}, as a function of t.
We have $A = 4I + N$, where $N = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$ and $N^2 = 0$. Hence

$$\begin{aligned} e^{tA} &= e^{4tI+tN} = e^{4t}e^{tN} \\ &= e^{4t}(I + tN) \\ &= e^{4t}\begin{pmatrix} 1+2t & -t \\ 4t & 1-2t \end{pmatrix} \end{aligned}$$

(d) (2 pts.) Find a matrix B with positive eigenvalues such that $B^2 = A$.
If $B = 2I + M$ with $M^2 = 0$, then $B^2 = 4I + 4M$, so this equals A if and only if $M = (1/4)N$. Thus

$$B = \begin{pmatrix} 2.5 & -.25 \\ 1 & 1.5 \end{pmatrix}$$

6. Consider the system of differential equations:

$$\begin{aligned} f' &= g \\ g' &= 2f - g \end{aligned}$$

(a) (2 pts.) Write this system as a vector-valued differential equation.

$$X' = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} X$$

(b) (3 pts.) Find a fundamental solution set for the equation in part (a).
The characteristic polynomial of this matrix A is

$$\lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2).$$

$$\begin{aligned} Eig_{-2}(A) = NS \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} &= span \begin{pmatrix} -1 \\ 2 \end{pmatrix} \\ Eig_1(A) = NS \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} &= span \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Hence a fundamental solution set is given by

$$e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, e^{-2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

(c) (2 pts.) Compute the Wronskian of your solution set.

$$\det \begin{pmatrix} e^t & -e^{-2t} \\ e^t & 2e^{-2t} \end{pmatrix} = 3e^{-t}.$$

(d) (3 pts.) Find a pair of functions f, g satisfying the original system and such that $f(0) = -1$ and $g(0) = 5$.
We must have

$$\begin{pmatrix} f \\ g \end{pmatrix} = ae^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + be^{-2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

So we need to find a, b such that

$$\begin{pmatrix} 1 \\ 5 \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

Thus $a = 1$ and $b = 2$, so $f(t) = e^t - 2e^{-2t}$ and $g(t) = e^t + 4e^{-2t}$

7. For each of the following matrices, sketch and describe the trajectories of the solutions to the differential equation $Y'(t) = AY(t)$. In particular, exhibit any asymptotes and/or invariant lines, draw arrows indicating the direction of the flow along the solution, and say whether the origin is a stable or unstable node, saddle point, etc.

   (a) (3 pts.) $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$
   This is an unstable node, and the trajectories are tangent to the y-axis. The axes are invariant.

   (b) (2 pts.) $A = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$
   This is a saddle point, with the axes as invariant lines asymptotes. It approaches the x-axis as time goes to infinity.

   (c) (2 pts.) $A = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$
   This is an unstable spiral point, and is spiralling counterclockwise out.

   (d) (3 pts.) $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$
   This is another saddle point. The x-axis is the positive eigenline and the span of $(1, 2)$ is the negative eigenline.

8. Suppose $Y''(x) = kY(x)$ and $Y(0) = 0$, $Y'(3) = 0$.

   (a) For which values of $k \in \mathbf{R}$ is there a nontrivial solution to this equation?

   (b) Explain (prove) why you have found all such k. (This part will count more than the actual answer.)

   (c) For each such k, give the corresponding solutions.

Case 1. (2 pts.) $k = 0$. Then $Y(x) = ax + b$ and $Y'(x) = a$. Hence the boundary value conditions imply that $b = 0$ and that $a = 0$, so Y is trivial.
Case 2. (2 pts.) $k = \lambda^2 > 0$. Then $Y(x) = ae^{\lambda x} + be^{-\lambda x}$ and $Y'(x) = a\lambda e^{\lambda x} - b\lambda e^{-\lambda x}$. The boundary value conditions then imply that $a + b = 0$, so $a = -b$ and that $a\lambda(e^{3\lambda} - e^{-3\lambda}) = 0$. This implies that $a = 0$, so Y is trivial.
Case 3. (3 pts.) $k = -\lambda^2 < 0$. Then $Y(x) = a\sin(\lambda x) + b\cos(\lambda x)$ and $Y'(x) = a\lambda\cos(\lambda x) - b\lambda\sin(\lambda x)$. The boundary conditions say that $b = 0$, and that $a\lambda\cos(\lambda 3) = 0$. This happens if and only if $3\lambda = \frac{n\pi}{2}$ for some integer n, i.e., iff $\lambda = \frac{n\pi}{6}$. Now we can answer the questions:

(a) (1 pt.) $k = \frac{-n^2\pi^2}{36}$ for some integer n.

(b) See above.

(c) (2 pts.) $Y(x) = a\sin(\frac{n\pi}{6})$ for some constant a.

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9. Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is even, periodic with period 2π, and satisfies $f(x) = \sin x$ for $x \in [0, \pi]$.

   (a) (3 pts.) Draw a sketch of the graph of f, labeling your axes carefully.

   (b) (5 pts.) Find a Fourier series which represents f. Explain how you find the coefficients. You may use one or more of the formulas at the end of the test to evaluate the coefficients if you like.
   This must be the *cosine* series expansion of the sine function. Thus it is given by $a_0/2+\sum_1^\infty a_n \cos(nx)$ with $a_n = \frac{2}{\pi}\int_0^\pi \sin(x)\cos(nx)dx$.
   Now using the formula at the end, we find that if $n \neq 1$,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^\pi (\sin((n+1)x) + \sin((1-n)x))\,dx \\ &= \frac{1}{\pi}\left(\frac{-1}{n+1}\cos((n+1)x) + \frac{-1}{1-n}\cos((1-n)x)\right)\Big|_0^\pi \\ &= \frac{-1}{\pi}\left(\frac{-1}{n+1}((-1)^{n+1}-1) - \frac{1}{n-1}((-1)^{n-1}-1)\right) \end{aligned}$$

   This is 0 if n is odd and is

$$\frac{2}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) = \frac{4}{\pi(1-n^2)}$$

   if n is even. Note that when $n = 1$, the integral is also zero. Hence the Fourier series is

$$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n>0,\text{even}} \frac{\cos(nx)}{n^2-1}$$

   (c) (2 pts.) Does the series converge to f at every point? Explain.
   Yes, because the function f and its derivative are piecewise continuous, and f is actually continuous.

10. Suppose that a bar of length π with thermal coefficient $\alpha^2 = 2$ in insulated on its surface except at the end points.

(a) (2 pts.) Write the differential equation governing heat diffusion in the bar described above.

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}.$$

(b) (3 pts.) If the ends of the bar are kept at 0^o and the initial temperature distribution is $u(x,0) = \sin(5x)$, find a formula for the temperature distribution at all times.

$$u(x,t) = e^{-2 \cdot 25t}\sin(5x) = e^{-50t}\sin(5x)$$

(c) (2 pts.) If instead one end is kept at π° and the other at $3\pi^\circ$, what is the limiting temperature distribution (steady state solution) of the temperature as $t \to \infty$? Verify directly that this limit distribution satisfies the equation in part (a).

$v(x,t) = \pi + 2x$. Evidently $\frac{\partial v}{\partial t} = 0 = 2\frac{\partial^2 v}{\partial x^2}$.

(d) (3 pts.) In the situation (c), assume again that the initial temperature is given by $u(x,0) = \sin(5x)$. Find a formula for the temperature distribution at time t. You may use one or more of the formulas listed at the end of the test.

Let $w := u - v$. Then w satisfies (a) but

$$w(x,0) = \sin(5x) - \pi - 2x.$$

We can then solve for w using the principle of superposition. Hence using the formulas at the end:

$$\begin{aligned} u(x,t) &= w(x,t) + v(x,t) \\ &= e^{-50t}\sin(5x) - 2\sum 1 + (-1)^{k+1}ke^{-2tk^2}\sin(kx) \\ + & \quad -4\sum\frac{(-1)^{k+1}}{k}e^{-2tk^2}\sin(kx) + \pi + 2x \end{aligned}$$

Formulas

$$
\begin{aligned}
1 &= 2\sum \frac{1+(-1)^{k+1}}{\pi k}\sin(kx) \qquad \text{for } 0 \le x \le \pi \\
x &= 2\sum \frac{(-1)^{k+1}}{k}\sin(kx) \qquad \text{for } 0 \le x \le \pi
\end{aligned}
$$

$$
\begin{aligned}
\sin(nx)\cos(x) &= \frac{\sin((1+n)x) + \sin((n-1)x)}{2} \\
\cos(nx)\sin(x) &= \frac{\sin((1+n)x) + \sin((1-n)x)}{2}
\end{aligned}
$$