Math 54 Professor K. A. Ribet

Midterm Exam April 3, 2007

This exam was an 80-minute exam. It began at 3:40PM. There were 4 problems, for which the point counts were 9, 6, 7 and 8. The maximum possible score was 30.

> Please put away all books, calculators, and other portable electronic devices—anything with an ON/OFF switch. You may refer to a single 2-sided sheet of notes. When you answer questions, explain in words what you are doing: your paper is your ambassador when it is graded. Correct answers without appropriate supporting work will be regarded with great skepticism. Incorrect answers without appropriate supporting work will receive no partial credit. Please write your name on each page of this exam. At the conclusion, please hand in your paper to your GSI.

1. *Let* $A = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -2 & -1 \\ 4 & 4 & 2 \end{bmatrix}$. *Calculate the characteristic polynomial of A. Determine bases for each of the eigenspaces of A. Decide whether or not the matrix A is diagonalizable.*

View the characteristic polynomial as a determinant and expand along the top row. If the variable is "t," you get t times the characteristic polynomial of the 2×2 matrix in the lower right-hand corner. This latter matrix has trace 0 and determinant 0, so its characteristic polynomial is t^2. Thus the characteristic polynomial of A is t^3, and 0 is the only eigenvalue. Clearly, the matrix is not diagonalizable because otherwise it would be the 0-matrix! The eigenspace corresponding to the eigenvalue 0 is the null space of A. By Gaussian elimination or calculation with equations, you see that the null space is 2-dimensional. One basis is the set consisting of the two vectors $(-1, 0, 2)$ and $(-1, 1, 0)$.

2. *Find numbers x and y so that the product* $\begin{bmatrix} -1 & 2 \\ -2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ *is as close as possible to the vector* $\begin{bmatrix} 35 \\ 35 \\ 35 \end{bmatrix}$.

Let $A = \begin{bmatrix} -1 & 2 \\ -2 & 3 \\ 1 & 1 \end{bmatrix}$. Then $\begin{bmatrix} x \\ y \end{bmatrix} = (A^T A)^{-1} A^T \begin{bmatrix} 35 \\ 35 \\ 35 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \end{bmatrix}$. All those 35's on the right-hand side were meant to be a clue that the correct values for x and y are whole numbers.

3. *Let A be a 3×3 matrix with eigenvalues 0, 1, 2 and corresponding eigenvectors v_0, v_1 and v_2.*

a. *Find bases for* $\mathrm{CS}(A)$ *and* $\mathrm{NS}(A)$.

The three eigenvectors are linearly independent (because they are associated with distinct eigenvalues), so they form a basis of $\mathbf{R}^3$. Every vector in $\mathbf{R}^3$ is of the form $av_0+bv_1+cv_2$. Now $A(av_0+bv_1+cv_2)=bv_1+2cv_2$, so CS($A$) is the span of the linearly independent vectors v_1 and v_2. Accordingly, v_1 and v_2 form a basis for CS(A). Further $A(av_0+bv_1+cv_2)=0$ if and only if $b=c=0$. Hence NS(A) is the span of v_0, and the single non-zero vector v_0 forms a basis for the null space.

b. *Find two different vectors x such that $Ax=2v_1+v_2$.*

You could take $x=2v_1+\frac{1}{2}v_2$ or $v_0+2v_1+\frac{1}{2}v_2$.

c. *Describe the set of vectors x for which $Ax=v_0$.*

It's the empty set. The expression bv_1+2cv_2 can never be v_0.

4. *Let A be an $n\times n$ matrix such that $A^2=A$.*

a. *Show that the non-zero elements of the column space of A are eigenvectors for A and that the non-zero elements of the null space of A are eigenvectors for A.*

The column space of A consists of vectors of the form Ax with $x\in\mathbf{R}^n$. If $v=Ax$, then $Av=A^2x=Ax=v$. Thus if v is non-zero, it's an eigenvector for the eigenvalue 1. If v is in the null space of A, $Av=0=0\cdot v$, so that v (when non-zero) is an eigenvector with eigenvalue 0.

b. *Let $v_1,\dots,v_r$ be a basis for the column space of A and let $w_1,\dots,w_s$ be a basis for the null space of A. (For simplicity, imagine that both the column space and the null space are non-zero.) Explain very briefly why $r+s=n$.*

This is the statement that the rank of an $m\times n$ matrix and the dimension of the null space of the matrix add up to n. This is one of the key theorems of the linear algebra portion of the course, and I was just trying to get you to state it.

c. *Prove that the $r+s$ vectors $v_1,\dots,v_r;w_1,\dots,w_s$ are linearly independent.*

In view of part (**b**), this statement means that the eigenvectors $v_1,\dots,v_r;w_1,\dots,w_s$ actually form a basis of $\mathbf{R}^n$. (There are n of them, and they're linearly independent.) How to prove the statement? Suppose that

$$0=(a_1v_1+\cdots+a_rv_r)+(b_1w_1+\cdots+b_sw_s).$$

Then

$$0=A(a_1v_1+\cdots+a_rv_r)+A(b_1w_1+\cdots+b_sw_s)=a_1v_1+\cdots+a_rv_r.$$

(Remember that $A(v_i)=v_i$ for all i and that $A(w_j)=0$ for all j.) Since the v_i are linearly independent, all the a_i are 0. Then $0=b_1w_1+\cdots+b_sw_s$; since the w_j are linearly independent, all the b_j are zero as well.