

**Math H53, Honors Multivariable Calculus (Kedlaya, fall 2002)**  
**Final exam, Friday, December 13, 12:30–3:30 PM, 60 Evans**

The only permitted aid is two  $8\frac{1}{2} \times 11$  sheets of paper (front and back). No other notes, calculator, or other assistance are permitted. Please write all answers on the exam sheet. Do not use a blue book. At the end of the exam, you will find an extra page for answers that do not fit on the problem pages, and an extra page for scratch work.

Overall course grades will be posted to BearFacts next week. Consult the course web page for information about obtaining your final exam grade.

There are nine questions (not ten as on the sample), and the maximum number of points is 200 (not 250 as on the sample).

Your name:

Your SID:

**Problem 1.** (20 points) Consider a circle of radius 1 rolling without slipping counterclockwise in the interior of a circle of radius 3 centered at the origin. At time  $t = 0$ , a marked point on the small circle lies on the point  $(3, 0)$  on the large circle. This next happens again at  $t = 2\pi$ .

- (a) Write parametric equations, in rectangular coordinates, for the location of the marked point. (Hint: first consider the center of the small circle.) (5 points)
- (b) Set up an integral to compute the arc length of the curve traced out by the marked point from  $t = 0$  to  $t = 2\pi$ . (5 points)
- (c) Evaluate the integral. (Hint: it may help to recall that  $1 - \cos 2\theta = 2 \sin^2 \theta$ .) (10 points)

**Problem 2.** (25 points) Define the points

$$P = (1, -1, 1), Q = (-1, 2, 1), R = (0, 1, 3), S = (-1, -1, -1).$$

- (a) Verify that  $P, Q, R, S$  do not lie in a plane. (10 points)
- (b) Find equations for the line  $L_1$  joining the midpoints of  $PQ$  and  $RS$ , and the line  $L_2$  joining the midpoints of  $PR$  and  $QS$ . (10 points)
- (c) Prove that  $L_1$  and  $L_2$  intersect. (Hint: you can do this without any computations. Think in terms of vectors to see what the common point is.) (5 points)

**Problem 3.** (20 points) Define the function  $f(x, y) = x^2 + 2xy^3 + y^4$ .

- (a) Find the critical points of  $f$ . (10 points)
- (b) Classify each critical point you found in (a) using the second derivative test. (10 points)

**Problem 4.** (15 points) Let  $S$  be the upper sheet of the hyperboloid  $z^2 = x^2 + y^2 + 1$ .

- (a) Use Lagrange multipliers to set up a system of equations whose solutions include the point(s) on  $S$  closest to  $(8/3, 16/3, 12)$ . (Hint: make sure your system has the same number of variables as equations.) (10 points)
- (b) Verify that the point  $(4, 8, 9)$  gives a solution to the equations in (a). (5 points)

**Problem 5.** (20 points) Let  $V$  be the portion of the ball  $x^2 + y^2 + z^2 \leq 1$  between the planes  $z = 0$  and  $z = 1/2$ .

- (a) Compute the volume of  $V$ . (10 points)
- (b) Set up integrals to compute the coordinates of the center of mass of  $V$ , assuming uniform density. (Hint: recall that the  $x$ -coordinate of the center of mass is the average value of  $x$  over  $V$ , and so on.) (5 points)
- (c) Evaluate the integrals in (b). (5 points)

**Problem 6.** (20 points) Let  $F$  be the region described by the inequalities

$$1 \leq xy^2 \leq 2, \quad 1 \leq x^2y^3 \leq 2.$$

- (a) Find a change of coordinates  $x = x(u, v)$ ,  $y = y(u, v)$  such that the region in the  $uv$ -plane corresponding to  $F$  is a rectangle with sides parallel to the axes. (5 points)
- (b) Write the area of  $F$  in terms of a double integral in the  $uv$ -plane. (10 points)
- (c) Evaluate the integral. (5 points)

**Problem 7.** (20 points) Let  $C$  be the spiral  $r = \theta$  between  $\theta = 0$  and  $\theta = a$ , for some  $a > 0$ .

- (a) Set up an integral to find the integral of  $xy$  over  $C$  with respect to arc length. Do not attempt to evaluate the integral. (10 points)
- (b) Write down a vector field  $\mathbf{F}$  (not depending on  $a$ ) such that  $\int_C \mathbf{F} \cdot d\mathbf{r}$  is equal to the integral in (a). (10 points)



**Problem 8.** (25 points) Let  $S$  be the portion of the hyperbolic paraboloid of one sheet  $z^2 = x^2 + y^2 - 1$  contained in the cylinder  $x^2 + z^2 \leq 3$ , oriented outward. Define the vector field

$$\mathbf{F} = \langle x^2, 2yz, -2xz - z^2 \rangle.$$

- (a) Find a vector field  $\mathbf{G}$  such that  $\mathbf{F} = \text{curl } \mathbf{G}$ . (Hint: there are actually a lot of possibilities. Try assuming  $\mathbf{G} = \langle P, Q, R \rangle$  with  $P$  a function of  $y$  alone.) (10 points)
- (b) Give a parametrization for each of the two components of the boundary of  $S$ . (Hint: first find the projection of either component onto the  $xy$ -plane, by eliminating  $z$  from the two equations that define the boundary.) (5 points)
- (c) Use Stokes' theorem to rewrite  $\iint_S \mathbf{F} \cdot d\mathbf{S}$  as a line integral. Do not evaluate the integral. (Hint: be careful about orientations!) (10 points)

**Problem 9.** (35 points) Define the vector field

$$\mathbf{F} = \left\langle \frac{yz}{x^2 + y^2 + z^2}, \frac{xz}{x^2 + y^2 + z^2}, \frac{-2xy}{x^2 + y^2 + z^2} \right\rangle.$$

- (a) Calculate  $\operatorname{div} F$ . (5 points)
- (b) Set up double integrals to calculate the integral  $\iint_S \mathbf{F} \cdot d\mathbf{S}$ , where  $S$  is the boundary of the cylindrical region satisfying the inequalities  $x^2 + y^2 \leq 1$  and  $-1 \leq z \leq 1$ , oriented outward. (This includes a lateral part and two disks.) (10 points)
- (c) Verify that the surface integral in (b) is zero. (Hint: use symmetry considerations.) (5 points)
- (d) Calculate the integral  $\iiint_T \mathbf{F} \cdot d\mathbf{S}$ , where  $T$  is the sphere of radius 1 centered at the origin. (Hint: do this without any further calculations.) (10 points)
- (e) Explain why the answer to (d) does not follow from (a) alone. (5 points)