1 Math 250A Midterm

Professor. David Eisenbud. Fall 2007
Each problem is worth 10 points. You have 80 minutes to complete the exam.

1.1

Let R be a commutative ring containing a field K. Suppose that R is 2-dimensional as a K-vector space, but not a field. Prove that $R \cong K \times K$ or $R \cong K[t]/(t^2)$ as rings.

1.2

Let G be the group of permutations of $1, 2, 3, 4, 5, 6, 7$, and let X be the $G - setG/H$, consisting of the cosets of H, where H is the subgroup generated by the cycle $(1, 2, 3, 4, 5, 6, 7)$. Show that there is a subset $Y \subset X$ (specifically, $\emptyset \subsetneq Y \subsetneq X$) such that for every $\sigma \in G$ either $\sigma(Y) = Y$ or $\sigma(Y) \cap Y = \emptyset$.

1.3

Define the coproduct of two objects in a category. Use the definition to prove that in the category of (not necessarily abelian) groups, the direct product $Z/2 \times Z/3$ is not equal to the coproduct of $Z/2$ and $Z/3$.

1.4

Give a definition of project module. Let K be a field and let $I = (x, y)$ be the ideal of $R = K[x, y]$ generated by the elements x and y. Show that I is not a projective R-module.

1.5

Classify the groups of order 4 and 8 up to isomorphism.