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Math 113, Section 4
Fall 2007

Solutions to the Midterm Exam 2

1. (5+4 points)

a) Give a definition of a ring.

b) Give a definition of a field.

(You do not need to explain any other words you use.)

Solution:
A ring $\langle R, +, \cdot \rangle$ is a set R with two binary operations $+$ and $\cdot$ such that $\langle R, + \rangle$ is an abelian group, $\cdot$ is associative, and the distributive laws hold.
A field is a commutative ring with unity $1 \neq 0$ such that each non-zero element is invertible.

2. (7 points)

For each of the following statements indicate whether it is true or false. You do not have to justify your answer.

A_n is a normal subgroup of S_n.

TRUE, since each subgroup of order 2 is normal or since $A_n = ker(sign)$.

Every subgroup of an abelian group is normal.

TRUE, since if G is abelian and H is a subgroup,then $aH = Ha$ for all $a \in G$.

If the commutator subgroup of a group is $\{e\}$, then G is abelian.

TRUE, since then, all commutators $aba^{-1}b^{-1}$ are equal to e, and it follows that $ab = ba$ for all elements a, b.

Every integral domain is a field.

FALSE, e.g. $\mathbb{Z}$ is an integral domain, but not a field.

A ring homomorphism is one-to-one if and only if its kernel is equal to $\{0\}$.

TRUE. We proved this for group homomorphisms. A ring homomorphism is also a group homomorphism of the additive groups of the rings.

The rings $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_6$ are isomorphic.

TRUE by the chinese remainder theorem.

The groups $\mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_8$ are isomorphic.

FALSE, for example by the classification of finitely generated abelian groups, or since the second group is cyclic, but not the first.

3. (6 points)

Let G be a group acting on a set X, and let $x \in X$. Prove that the isotropy subgroup $G_x = \{g \in G \mid gx = x\}$ is a group.

Solution:
We prove that G_x is a subgroup of G. If $g, h \in G_x$, then $gh \in G_x$, since in this case $gh \cdot x = g(hx) = gx = x$ by definition of a group action. $e \in G_x$ since $ex = x$ by definition of a group action. If $g \in G_x$, then $g^{-1} \in G_x$: from $gx = x$ it follows that $x = g^{-1}gx = g^{-1}x$. So G_x is a subgroup of G.

4. (3+4 points)

a) Determine the remainder of the division of 6^{3000} by 61.

b) Determine the remainder of the division of 7^{3217} by 34.

Solution:
a) 61 is prime, so we use the little theorem of Fermat to see that $6^{60} \equiv 1 \pmod{61}$. It follows that $6^{3000} = (6^{60})^{50}$ has remainder 1 when divided by 61.
b) We use Euler's theorem: $\phi(34) = 16$, and since $gcd(7, 34) = 1$, it follows that $7^{16} \equiv 1 \pmod{34}$. It follows that $7^{3217} = (7^{16})^{201} \cdot 7$ has remainder 7 when divided by 34.

5. (6 points)

Let $(a_1\ a_2\ \ldots a_k) \in S_n$ be a k-cycle, and let $\sigma \in S_n$ be an arbitrary permutation. Show that $\sigma(a_1\ a_2\ \ldots a_k)\sigma^{-1} = (\sigma(a_1)\ \sigma(a_2)\ \ldots \sigma(a_k))$.

Solution:

$$(\sigma(a_1\ a_2\ \ldots a_k)\sigma^{-1})(\sigma(a_i)) = (\sigma(a_1\ a_2\ \ldots a_k))(a_i) = \sigma(a_{i+1})$$

for $i < k$, and

$$(\sigma(a_1\ a_2\ \ldots a_k)\sigma^{-1})(\sigma(a_k)) = (\sigma(a_1\ a_2\ \ldots a_k))(a_k) = \sigma(a_1).$$

If $j \notin \{\sigma(a_1), \ldots \sigma(a_k)\}$, then $\sigma^{-1}(j) \notin \{a_1, \ldots a_k\}$, and

$$(\sigma(a_1\ a_2\ \ldots a_k)\sigma^{-1})(j) = (\sigma(a_1\ a_2\ \ldots a_k))(\sigma^{-1}(j)) = \sigma(\sigma^{-1}(j)) = j.$$

It follows that $\sigma(a_1\ a_2\ \ldots a_k)\sigma^{-1} = (\sigma(a_1)\ \sigma(a_2)\ \ldots \sigma(a_k))$.

6. (6+3 points)

The dihedral group $D_4 = \{e, (1\,2\,3\,4), (1\,3)(2\,4), (1\,4\,3\,2), (1\,4)(2\,3), (1\,3), (2\,4), (1\,2)(3\,4)\}$ is a subgroup of S_4 of order 8.

a) Show that $H = \langle(1\ 3)(2\ 4)\rangle$ is a normal subgroup of D_4.

b) Find an abelian group isomorphic to D_4/H.

Solution:
a) We compute the left and right cosets: The left cosets are $H = \{e, (1\,3)(2\,4)\}, (1\,2\,3\,4)H = \{(1\ 2\ 3\ 4), (1\ 4\ 3\ 2)\}, (1\ 3)H = \{(1\ 3), (2\ 4)\}, (1\ 4)(2\ 3)H = \{(1\ 4)(2\ 3), (1\ 2)(3\ 4)\}$. The right cosets are $H = \{e, (1\ 3)(2\ 4)\}, H(1\ 2\ 3\ 4) = \{(1\ 2\ 3\ 4), (1\ 4\ 3\ 2)\}, H(1\ 3) = \{(1\ 3), (2\ 4)\}, H(1\ 4)(2\ 3) = \{(1\ 4)(2\ 3), (1\ 2)(3\ 4)\}$. Since right and left cosets are equal, H is a normal subgroup of D_4.
b) D_4/H has 4 elements, so we only have to check whether the group is isomorphic to $\mathbb{Z}_4$ or to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since $(1\ 2\ 3\ 4)H(1\ 2\ 3\ 4)H = (1\ 3)(2\ 4)H = H, (1\ 3)H(1\ 3)H = eH = H, (1\ 4)(2\ 3)H(1\ 4)(2\ 3)H = eH = H$, we see that D_4/H is not cyclic, hence isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

7. (6 points)

Let $\phi : G \to G'$ be a group homomorphism between a group G of order 36 and a group G' of order 55. Use the homomorphism theorem and the theorem of Lagrange to prove that ϕ is the trivial homomorphism.

Solution:
By the homomorphism theorem, the groups $G/Ker(\phi)$ and $\phi(G)$ are isomorphic. The order of $G/Ker(\phi)$ is the index of $Ker(\phi)$ in G, so a divisor of 36 by the theorem of Lagrange. $\phi(G)$ is a subgroup of G', so its order is a divisor of 55 by the theorem of Lagrange. Since $G/Ker(\phi)$ and $\phi(G)$ are isomorphic, they have the same order, which must be a common divisor of 36 and 55. But 36 and 55 are relatively prime, so $G/Ker(\phi)$ and $\phi(G)$ both have order 1. It follows that $\phi(G) = \{e\}$, so ϕ maps each element of G to e.