(20) 1. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 1 & 1 & 2 \\ 4 & 1 & 0 & 1 \end{bmatrix}$$

Find *rank* A, a basis for *Col* A and a basis for *Row* A.

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 1 & 1 & 2 \\ 4 & 1 & 0 & 1 \end{bmatrix} R_3 - 2R_2 \quad \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 1 & 1 & 2 \\ 0 & -1 & -2 & -3 \end{bmatrix} \begin{matrix} R_2 - R_1 \\ R_3 + R_1 \end{matrix} \quad \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank $A = 2$ ✓

because there are two columns with pivots

basis for Col A = $\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ✓

$$A^T = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{matrix} \\ \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \quad \begin{bmatrix} 0 & 2 & 4 \\ 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix} \begin{matrix} \tfrac{1}{2}R_1 \\ \\ \\ R_4 - R_3 \end{matrix} \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix} R_3 + R_2 \quad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_2 - R_1$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

basis for Row A = $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$ ✓

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(20) 6. Mark each statement True or False. Justify your answers.

a) If A and B are $n \times n$ matrices so that AB is invertible then BA is invertible.

True if $(AB)^{-1}$ exists then $\det(AB) \neq 0$

since determinants are multiplicative $\det(AB) = \det A \det B$

so $\det A \det B \neq 0$ neither $\det A$ nor $\det B$ is 0

so $\det B \det A$ is non zero as well and $\det BA \neq 0$

so BA^{-1} exists

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b) Any 3 linearly independent vectors form a basis in R^3 form a basis.

~~False~~

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

are 3 linearly independent vectors that form a basis outside R^3

(20) 5. Consider the set W of all polynomials p in P_3 with $p(1) = 0$.
a) Show that W is a subspace of P_3.

$w = \{ax^3 + bx^2 + cx + d\}$ $\quad 0 = a+b+c+d$

✓ 13

to be a subspace of P_3

W must
- be a subset of P_3 ✓ it has all polynomials p in P_3
- must contain the zero polynomial ✓ $p(1) = 0$
- must be closed under addition ✓

any $ax^3 + bx^2 + cx + d + a'x^3 + b'x^2 c'x + d'$

$a+b+c+d = 0$
$+a'+b'+c'+d' = 0$
0

$= (a+a')x^3 + (b+b')x^2 + (c+c')x + d+d'$ $\quad (a+a') + (b+b') + (c+c') + d+d'$

so the sum of the two vectors is still in the set w

- must be closed under addition ✓

$2(a+b+c+d) = 2(0)$ so the resulting coefficients still satisfy the requirement to be in w

b) Find a basis in W.

$\boxed{x^3, \; -x^2, \; x, \; -1}$

X 0

these have coefficients such that

$a+b+c+d = 0$
$1-1+1-1 = 0$

also because the standard basis for P_3 is $x^3 \; x^2 \; x \; 1$

taking the negative of two elements should not take away their linear independence

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(20) 2. **Problem 2.** Compute (or if undefined say so, explaining why)

5 a) A^{-1}, $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}$

A^{-1} is undefined because A is not a square matrix and is is not invertible

5 b) A^{2008}, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$1 \to 3$
$2 \to 1$
$3 \to 2$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

returns to itself after multiplication by itself three times so $A^4, A^7, A^{10} = A$

$2008 \div 3 = 669$ R1

2008
18
20
28
27
1

$A^{2007} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

5 c) $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} [1\ 2\ 4]$

$\begin{bmatrix} 1\cdot1 & 1\cdot2 & 1\cdot4 \\ 2\cdot1 & 2\cdot2 & 2\cdot4 \\ 4\cdot1 & 4\cdot2 & 4\cdot4 \end{bmatrix}$

3×1, 1×3
$\downarrow$
3×3

$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}$

5 d) det $\begin{bmatrix} 7 & 0 & 0 & 4 & 0 \\ 1 & 1 & 2 & 5 & 0 \\ 1 & 4 & 7 & 5 & 2 \\ 3 & 0 & 0 & 2 & 0 \\ 2 & 0 & 1 & 1 & 0 \end{bmatrix}$

$+\ -\ +\ -\ +$

$7 \begin{vmatrix} 1 & 2 & 5 & 0 \\ 4 & 7 & 5 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{vmatrix} - 0 + 0 - 4 \begin{vmatrix} 1 & 1 & 2 & 0 \\ 1 & 4 & 7 & 2 \\ 3 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{vmatrix} + 0$

$7\left(2 \begin{vmatrix} 1 & 2 & 5 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix}\right) - 4\left(2 \begin{vmatrix} 1 & 1 & 2 \\ 3 & 0 & 0 \\ 2 & 0 & 1 \end{vmatrix}\right)$

$7\left(2\left(-2 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}\right)\right) - 4\left(2\left(-3 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}\right)\right) = -28(1) + 24(1) = \boxed{-4}$

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20) 3. a) State Cramer's rule.

$$x_i = \frac{\det(A_i)}{\det(A)}$$

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix}\text{cofactors of } A \\ \text{with sign.}\end{pmatrix}^T$$

A_i is A with ith column replaced by b in $Ax=b$

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Cramer's rule shows us why A^{-1} is undefined when $\det A=0$ and also tells us that the elements of A^{-1} will be integers when $\det A=1$ if the entries of A were integers.

These are consequences of Cramer's Rule, <u>not</u> Cramer's rule.

b) Find the area of the triangle with vertices $(3,2)$, $(7,4)$ and $(4,5)$.

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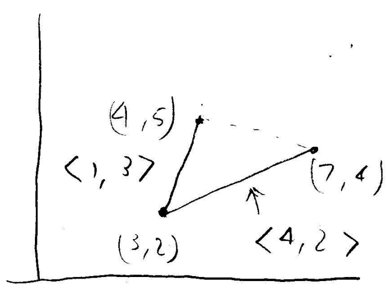

$$\frac{\text{abs}\left(\begin{vmatrix}1 & 3\\ 4 & 2\end{vmatrix}\right)}{2} = \text{abs}(2-12) = \frac{10}{2} = 5$$

$$\boxed{\text{Area} = 5}$$ ✓

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(20) 4. Mark each statement True or False. Justify your answers.

a) $AB = BA$ for all square matrices A, B.

False for example

$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

square matrices

$BA = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ clearly $AB \neq BA$ ✓

b) The set V of all 3×4 matrices is a vector space of dimension 12.

false to be a vector space of dimension 12 the set V would have to have 12 basis vectors however V being all 3X4 matrices has at most a rank of 3 meaning it has at most 3 basis elements not 12