Midterm I, October 5

Work each problem on a separate sheet of paper. Be sure to put your name, your section number, and your GSI's name on each sheet of paper. Also, at the top of the page, in the center, write the problem number, and be sure to put the pages in order. Write clearly: explanations (with complete sentences when appropriate) will help us understand what you are doing. Note that there are problems on the back of this sheet, for a total of four problems.

1. Consider the following system of linear equations:

$$\begin{aligned} x_1 - x_2 + x_3 &= 5 \\ x_3 - 2x_1 + x_4 &= 2. \end{aligned}$$

   (a) (5 pts) Write the system in matrix form. (You may either write an augmented matrix or a matrix equation.)

   (b) (5 pts) How many solutions does the system have? Explain your answer. You do not need to find the solutions explicitly.

2. Compute the following matrix operations if they are defined. If not, explain why not.

   (a) (3 pts) $\begin{pmatrix} 1 & 2 \\ 3 & -2 \end{pmatrix} + 3 \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$

   (b) (3 pts) $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}^{-1}$

   (c) (3 pts) $\begin{pmatrix} 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

   (d) (3 pts) $\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

   (e) (3 pts) $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}$

3. Let

$$A := \begin{pmatrix} 1 & -1 & 2 & 3 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & -2 \end{pmatrix}$$

(a) (5 pts) Find the reduced row echelon form of A.

(b) (5 pts) What is the rank of A? What is the dimension of the kernel of the linear transformation T_A corresponding to A? Explain.

(c) (5 pts) Find a basis of the image of the linear transformation T_A.

4. Let P_3 denote the vector space of polynomials p of degree at most three. You may use the fact that $\beta := (x^0, x^1, x^2, x^3)$ is a basis of P_3. Note that x^0 is the constant function whose value is always 1. Let T be the linear transformation from P_3 to $\mathbf{R}^2$ taking p to $\begin{pmatrix} p(-1) \\ p(1) \end{pmatrix}$.

(a) (5 pts) What is the dimension of P_3? Explain, recalling the definition of dimension.

(b) (5 pts) Find a basis for the kernel of T. You do not need to prove that your answer is correct.

(c) (5pts) Find the matrix $[T]_\gamma^\beta$ for T with respect to the basis β for P_3 and the standard basis $\gamma := (e_1, e_2)$ for $\mathbf{R}^2$.