Math 55 Second Midterm 6 Apr 2004

NAME (1 pt): ______________________________

TA (1 pt): ______________________________

Name of Neighbor to your left (1 pt): ______________________________

Name of Neighbor to your right (1 pt): ______________________________

Instructions: This is a closed book, closed notes, closed calculator, closed computer, closed network, open brain exam.

You get one point each for filling in the 4 lines at the top of this page. All other questions are worth 12 points.

Write all your answers on this exam. If you need scratch paper, ask for it, write your name on each sheet, and attach it when you turn it in (we have a stapler).

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Total	

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Question 1. (12 points) Prove by induction that $15|(10 \cdot 2^n + 2 \cdot 11^n + 3 \cdot 6^n)$ for integers $n \geq 0$. You must use proof by induction. Hint: Write $10 \cdot 2^{n+1} + 2 \cdot 11^{n+1} + 3 \cdot 6^{n+1} = 2 \cdot (10 \cdot 2^n + 2 \cdot 11^n + 3 \cdot 6^n) + 18 \cdot 11^n + 12 \cdot 6^n$ and then use induction to prove that $15|(18 \cdot 11^n + 12 \cdot 6^n)$.

Answer: *The base case ($n = 0$) holds because* $15|10 + 2 + 3 = 15$. *For the induction step consider* $10 \cdot 2^{n+1} + 2 \cdot 11^{n+1} + 3 \cdot 6^{n+1} = 2 \cdot (10 \cdot 2^n + 2 \cdot 11^n + 3 \cdot 6^n) + 18 \cdot 11^n + 12 \cdot 6^n$. *To use induction we need to show that* $15|18 \cdot 11^n + 12 \cdot 6^n$ *or* $5|6 \cdot 11^n + 4 \cdot 6^n$, *which requires another induction proof.*

The base case for this ($n = 0$) holds because $5|6 + 4 = 10$. *For the induction step consider* $6 \cdot 11^{n+1} + 4 \cdot 6^{n+1} = 11 \cdot (6 \cdot 11^n + 4 \cdot 6^n) - 20 \cdot 6^n$. *Clearly* $5|20$ *so* $5|20 \cdot 6^n$ *so by induction* $5|6 \cdot 11^n + 4 \cdot 6^n$.

Returning to the first proof, we have $15|18 \cdot 11^n + 12 \cdot 6^n$ *so that by induction* $15|10 \cdot 2^n + 2 \cdot 11^n + 3 \cdot 6^n$.

Question 1. (12 points) Prove by induction that $14|(7 \cdot 3^n + 6 \cdot 4^n + 11^n)$ for integers $n \geq 0$. You must use proof by induction. Hint: Write $7 \cdot 3^{n+1} + 6 \cdot 4^{n+1} + 11^{n+1} = 3 \cdot (7 \cdot 3^n + 6 \cdot 4^n + 11^n) + 6 \cdot 4^n + 8 \cdot 11^n$ and then use induction to prove that $14|(6 \cdot 4^n + 8 \cdot 11^n)$.

Answer: *The base case* ($n = 0$) *holds because* $14|7 + 6 + 1 = 14$. *For the induction step consider* $7 \cdot 3^{n+1} + 6 \cdot 4^{n+1} + 11^{n+1} = 3 \cdot (7 \cdot 3^n + 6 \cdot 4^n + 11^n) + 6 \cdot 4^n + 8 \cdot 11^n$. *To use induction we need to show that* $14|6 \cdot 4^n + 8 \cdot 11^n$ *or* $7|3 \cdot 4^n + 4 \cdot 11^n$, *which requires another induction proof.*

The base case for this ($n = 0$) *holds because* $7|3 + 4 = 7$. *For the induction step consider* $6 \cdot 4^{n+1} + 8 \cdot 11^{n+1} = 4 \cdot (6 \cdot 4^n + 8 \cdot 11^n) + 56 \cdot 11^n$. *Clearly* $7|56$ *so* $7|56 \cdot 11^n$ *so by induction* $7|3 \cdot 4^n + 4 \cdot 11^n$.

Returning to the first proof, we have $14|6 \cdot 4^n + 8 \cdot 11^n$ *so that by induction* $14|7 \cdot 3^n + 6 \cdot 4^n + 11^n$.

Question 1. (12 points) Prove by induction that $6|(3 \cdot 3^n + 2 \cdot 4^n + 7^n)$ for integers $n \geq 0$. You must use proof by induction. Hint: Write $3 \cdot 3^{n+1} + 2 \cdot 4^{n+1} + 7^{n+1} = 3 \cdot (3 \cdot 3^n + 2 \cdot 4^n + 7^n) + 2 \cdot 4^n + 4 \cdot 7^n$ and then use induction to prove that $6|(2 \cdot 4^n + 4 \cdot 7^n)$.

Answer: *The base case ($n = 0$) holds because $6|2 + 3 + 1 = 6$. For the induction step consider $3 \cdot 3^{n+1} + 2 \cdot 4^{n+1} + 7^{n+1} = 3 \cdot (3 \cdot 3^n + 2 \cdot 4^n + 7^n) + 2 \cdot 4^n + 4 \cdot 7^n$. To use induction we need to show that $6|2 \cdot 4^n + 4 \cdot 7^n$ or $3|4^n + 2 \cdot 7^n$, which requires another induction proof.*

The base case for this ($n = 0$) holds because $3|1 + 2 = 3$. For the induction step consider $4^{n+1} + 2 \cdot 7^{n+1} = 4 \cdot (4^n + 2 \cdot 7^n) + 6 \cdot 7^n$. Clearly $3|6$ so $3|6 \cdot 7^n$ so by induction $3|4^n + 2 \cdot 7^n$.

Returning to the first proof, we have $6|2 \cdot 4^n + 4 \cdot 7^n$ so that by induction $6|3 \cdot 3^n + 2 \cdot 4^n + 7^n$.

Question 2. (12 points) From a group of 2 freshman (Joey and Zoe), 3 sophomores (Albert, Winny, and Paul), 4 juniors (Betty, Cal, Dean, and Elly), and 5 seniors (Fred, Greta, Harry and Isaac) we wish to arrange a portrait of 10 of them standing in a row. How many ways are there to arrange the students under the following conditions (each set of conditions is independent of the others)? You may leave expressions like $C(8,4)$ or 56! unsimplified in your answers.

1. No conditions: any 10 students may appear in any order.

   Answer: $P(2+3+4+5,10) = P(14,10) = 14!/4! = 3632428800.$

2. There must be exactly 4 seniors and 3 juniors in the portrait.

   Answer: *#(ways to pick 4 locations out of 10 for seniors) * #(ways to arrange 4 seniors out of 5 in these locations) * #(ways to pick 3 locations out of remaining 6 for juniors) * #(ways to arrange 3 juniors out of 4 in these locations) * #(ways to fill in remaining 3 locations) = C(10,4) * P(5,4) * C(6,3) * P(4,3) * P(2 + 3, 3) = 725760000.*

   *Here is another way: #(ways to choose 4 out of 5 senior) * #(ways to choose 3 out of 4 juniors) * #(ways to choose 3 out of 2 freshmen + 3 sophomores) * #(ways to order all 10 students) = C(5,4)*C(4,3)*C(5,3)*10!*

3. There must be a group of 7 consecutive students in a row of the types (freshman, sophomore, junior, senior, junior, sophomore, freshman)

   Answer: *#(places to start the group of 7 students) * #(ways to pick the 2 freshman in the group of 7) * #(ways to pick the 2 sophomores in the group of 7) * #(ways to pick the 2 juniors in the group of 7) * #(ways to pick the 1 senior in the group of 7) * #(ways to pick the remaining 3 slots) = 4 * P(2,2) * P(3,2) * P(4,2) * P(5,1) * P(2+3+4+5-7,3) = 604800.*

Question 2. (12 points) From a group of 3 goats (Billy, Willy and Silly), 3 cows (Eeny, Meeny and Miny), 5 pigs (Blimpy, Wimpy, Gimpy, Jimpy and Dimpy) and 5 reindeer (Donner, Dancer, Prancer, Blitzen and Comet) we wish to arrange a portrait of 11 of them standing in a row. How many ways are there to arrange the animals under the following conditions (each set of conditions is independent of the others)? You may leave expressions like $C(8,4)$ or 56! unsimplified in your answers.

1. No conditions: any 11 animals may appear in any order.

   Answer: $P(3+3+5+5, 11) = P(16, 11) = 16!/5! = 174356582400.$

2. There must be exactly 4 pigs and 2 cows in the portrait.

   Answer: *#(ways to pick 4 locations out of 11 for pigs) * #(ways to arrange 4 pigs out of 5 in these locations) * #(ways to pick 2 locations out of remaining 7 for cows) * #(ways to arrange 2 cows out of 3 in these locations) * #(ways to fill in remaining 5 locations) = C(11,4) * P(5,4) * C(7,2) * P(3,2) * P(3 + 5, 5) = 33530112000.*

   *Here is another way: #(ways to choose 4 out of 5 pigs) * #(ways to choose 2 out of 3 cows) * #(ways to choose 5 out of 3 goats + 5 reindeer) * #(ways to order all 11 animals) = C(5,4)*C(3,2)*C(8,5)*11!*

3. There must be a group of 8 consecutive animals in a row of the types (goat, cow, pig, reindeer, reindeer, pig, cow, goat).

   Answer: *#(places to start the group of 8 animals) * #(ways to pick the 2 goats in the group of 8) * #(ways to pick the 2 cows in the group of 8) * #(ways to pick the 2 pigs in the group of 8) * #(ways to pick the 2 reindeer in the group of 8) * #(ways to pick the remaining 3 slots) = 4 * P(3,2) * P(3,2) * P(5,2) * P(5,2) * P(3+3+5+5-8,3) = 19353600.*

Question 2. (12 points) From a group of 3 Bears (Left, Right and Center), 4 Cardinals (Forward, Back, Guard, Goalie), 4 Bruins (Harold, Hanna, Hattie and Hagrid), and 5 Trojans (Metis, Midas, Memnon, Maia, and Milo) we wish to arrange a portrait of 12 of them standing in a row. How many ways are there to arrange the athletes under the following conditions (each set of conditions is independent of the others)? You may leave expressions like $C(8,4)$ or 56! unsimplified in your answers.

1. No conditions: any 12 athletes may appear in any order.

   Answer: $P(3+4+4+5,12) = P(16,12) = 16!/4! = 871782912000$.

2. There must be exactly 3 Cardinals and 4 Trojans in the portrait.

   Answer: *#(ways to pick 3 locations out of 12 for Cardinals) * #(ways to arrange 3 Cardinals out of 4 in these locations) * #(ways to pick 4 locations out of remaining 9 for Trojans) * #(ways to arrange 4 Trojans out of 5 in these locations) * #(ways to fill in remaining 5 locations) = C(12,3) * P(4,3) * C(9,4) * P(5,4) * P(3 + 4, 5) = 201180672000.*

   *Here is another way: #(ways to choose 3 out of 4 Cardinals) * #(ways to choose 4 out of 5 Trojans) * #(ways to choose 5 out of 3 Bears + 4 Bruins) * #(ways to order all 12 athletes) = C(4,3)*C(5,4)*C(7,5)*12!*

3. There must be a group of 9 consecutive athletes in a row of the types (Bear, Cardinal, Bruin, Trojan, Bear, Trojan, Bruin, Cardinal, Bear).

   Answer: *#(places to start the group of 9 athletes) * #(ways to pick the 3 Bears in the group of 9) * #(ways to pick the 2 Cardinal in the group of 9) * #(ways to pick the 2 Bruins in the group of 9) * #(ways to pick the 2 Trojans in the group of 9) * #(ways to pick the remaining 3 slots) = 4 * P(3,3) * P(4,2) * P(4,2) * P(5,2) * P(3+4+4+5-9,3) = 14515200.*

Question 3. (12 points) Find the smallest positive integers x and y simultaneously satisfying the following 4 equations: For full credit, you must explain your reasoning fully.

$$
\begin{aligned}
3x + 2y &\equiv 3 \bmod 7 \\
2x + 5y &\equiv 5 \bmod 7 \\
7x + 14y &\equiv 6 \bmod 11 \\
5x + 3y &\equiv 4 \bmod 11
\end{aligned}
$$

Hint: Start by solving the first two equations simultaneously for $x \bmod 7$ and $y \bmod 7$, and then the last two equations simultaneously for $x \bmod 11$ and $y \bmod 11$.

Answer: *Call the equations EQ1, EQ2, EQ3 and EQ4.*

Adding EQ1+EQ2 yields $5x \equiv 1 \bmod 7$ *or (multiplying by 3)* $x \equiv 3 \bmod 7$. *Plugging into EQ1 yields* $9 + 2y \equiv 3 \bmod 7$ *or* $2y \equiv 1 \bmod 7$ *or (multiplying by 4)* $y \equiv 4 \bmod 7$.

Subtracting EQ3-EQ4 yields $2x \equiv 2 \bmod 11$ *or* $x \equiv 1 \bmod 11$. *Plugging into EQ4 yields* $5 + 3y \equiv 4 \bmod 11$ *or* $3y \equiv 10 \bmod 11$ *or (multiplying by 4)* $y \equiv 7 \bmod 11$.

Solving $x \equiv 3 \bmod 7$ *and* $x \equiv 1 \bmod 11$ *with the Chinese Remainder Theorem yields the smallest positive solution* $x = 45$.

Solving $y \equiv 4 \bmod 7$ *and* $y \equiv 7 \bmod 11$ *with the Chinese Remainder Theorem yields the smallest positive solution* $y = 18$.

Question 3. (12 points) Find the smallest positive integers x and y simultaneously satisfying the following 4 equations: For full credit, you must explain your reasoning fully.

$$\begin{aligned} 4x+2y &\equiv 1 \bmod 7 \\ 3x+4y &\equiv 4 \bmod 7 \\ 8x+2y &\equiv 8 \bmod 9 \\ 6x+11y &\equiv 5 \bmod 9 \end{aligned}$$

Hint: Start by solving the first two equations simultaneously for $x \bmod 7$ and $y \bmod 7$ and then the last two equations simultaneously for $x \bmod 9$ and $y \bmod 9$.

Answer: *Call the equations EQ1, EQ2, EQ3 and EQ4.*

Adding EQ1+EQ2 yields $6y \equiv 5 \bmod 7$ *or (multiplying by -1)* $y \equiv 2 \bmod 7$. *Plugging into EQ2 yields* $3x+8 \equiv 4 \bmod 7$ *or* $3x \equiv 3 \bmod 7$ *or* $x \equiv 1 \bmod 7$.

Subtracting EQ4-EQ3 yields $-2x \equiv -3 \bmod 9$ *or* $2x \equiv 3 \bmod 9$ *or (multiplying by 5)* $x \equiv 6 \bmod 9$. *Plugging into EQ3 yields* $48+2y \equiv 8 \bmod 9$ *or* $2y \equiv 5 \bmod 9$ *or (multiplying by 5)* $y \equiv 7 \bmod 9$.

Solving $x \equiv 1 \bmod 7$ *and* $x \equiv 6 \bmod 9$ *with the Chinese Remainder Theorem yields the smallest positive solution* $x = 15$.

Solving $y \equiv 2 \bmod 7$ *and* $y \equiv 7 \bmod 9$ *with the Chinese Remainder Theorem yields the smallest positive solution* $y = 16$.

Question 3. (12 points) Find the smallest positive integers x and y simultaneously satisfying the following 4 equations: For full credit, you must explain your reasoning fully.

$$
\begin{aligned}
5x + 4y &\equiv 8 \bmod 9 \\
4x + y &\equiv 3 \bmod 9 \\
2x + 4y &\equiv 4 \bmod 11 \\
3x + 5y &\equiv 4 \bmod 11
\end{aligned}
$$

Hint: Start by solving the first two equations simultaneously for $x \bmod 9$ and $y \bmod 9$ and then the last two equations simultaneously for $x \bmod 11$ and $y \bmod 11$.

Answer: *Call the equations EQ1, EQ2, EQ3 and EQ4.*

Adding EQ1+EQ2 yields $9x + 5y \equiv 11 \bmod 9$ *or* $5y \equiv 2 \bmod 9$ *or (multiplying by 2)* $y \equiv 4 \bmod 9$*. Plugging this into EQ1 yields* $5x+16 \equiv 8 \bmod 9$ *or* $5x \equiv 1 \bmod 9$ *or (multiplying by 2)* $x \equiv 2 \bmod 9$*.*

Subtracting EQ4-EQ3 yields $x + y \equiv 0 \bmod 11$ *or* $x \equiv -y \bmod 11$*. Plugging into EQ3 yields* $2y \equiv 4 \bmod 11$ *or* $y \equiv 2 \bmod 11$*. Plugging into EQ3 yields* $2x + 8 \equiv 4 \bmod 11$ *or* $2x \equiv 7 \bmod 11$ *or (multiplying by 6)* $x \equiv 42 \equiv 9 \bmod 11$*.*

Solving $x \equiv 2 \bmod 9$ *and* $x \equiv 9 \bmod 11$ *with the Chinese Remainder Theorem yields the smallest positive solution* $x = 20$*.*

Solving $y \equiv 4 \bmod 9$ *and* $y \equiv 2 \bmod 11$ *with the Chinese Remainder Theorem yields the smallest positive solution* $y = 13$*.*

Question 4. (12 points) Each person in a group of 24 people chooses 12 other people in the group and sends each one a letter. Prove that some pair of people must send each other letters.

Answer: *Number the people in the group from 1 to 24. There are C(24,2) = 24*23/2 possible pairs of letter writers* $\{i, j\}$, *where* $1 \leq i \leq 24$, $1 \leq j \leq 24$, $i \neq j$.

*A total of 24*12 letters are written. If i writes a letter to j, or j writes a letter to i, put the letter (pigeon) into the "hole"* $\{i, j\}$. *Since* $12 > 23/2$, *the number of pigeons 24*12 exceeds the number of holes 24*23/2, and at least two pigeons get put into one hole, that is two people send each other letters.*

Question 4. (12 points) Each person in a group of 26 people chooses 13 other people in the group and sends each one a letter. Prove that some pair of people must send each other letters.

Answer: *Number the people in the group from 1 to 26. There are C(26,2) = 26*25/2 possible pairs of letter writers* $\{i, j\}$*, where* $1 \leq i \leq 26$*,* $1 \leq j \leq 26$*,* $i \neq j$*.*

*A total of 26*13 letters are written. If i writes a letter to j, or j writes a letter to i, put the letter (pigeon) into the "hole"* $\{i, j\}$*. Since* $13 > 25/2$*, the number of pigeons 26*13 exceeds the number of holes 26*25/2, and at least two pigeons get put into one hole, that is two people send each other letters.*

Question 4. (12 points) Each person in a group of 28 people chooses 14 other people in the group and sends each one a letter. Prove that some pair of people must send each other letters.

Answer: *Number the people in the group from 1 to 28. There are C(28,2) = 28*27/2 possible pairs of letter writers* $\{i, j\}$*, where* $1 \leq i \leq 28$, $1 \leq j \leq 28$, $i \neq j$

*A total of 28*14 letters are written. If i writes a letter to j, or j writes a letter to i, put the letter (pigeon) into the "hole"* $\{i, j\}$*. Since* $14 > 27/2$*, the number of pigeons 28*14 exceeds the number of holes 28*27/2, and at least two pigeons get put into one hole, that is two people send each other letters.*