Math 110-S1 Spring 2008

Name:

SID:

Midterm 1

Write your name and SID on the front of your exam. You must ***JUSTIFY*** *your answers, so show your work. Partial credit will be awarded even if answers are incorrect. No notes, books, or calculators. Good luck!*

1. Define $T\colon \mathrm{M}_{2\times 2}(\mathbb{R}) \to \mathrm{M}_{2\times 2}(\mathbb{R})$ as $T(A) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} A$.

a. (5 pts.) Prove T is linear and compute $[T]_\beta$, where β is the standard basis of $\mathrm{M}_{2\times 2}(\mathbb{R})$.
b. (5 pts.) Give bases for $\mathrm{N}(T)$ and $\mathrm{R}(T)$.
c. (5 pts.) Let $\beta' = \{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\}$. Compute $[T]_{\beta'}$.
d. (5 pts.) Find an invertible matrix Q such that $[T]_{\beta'} = Q^{-1}[T]_\beta Q$. You do NOT need to compute Q^{-1}.

SOLUTION:
a. Linearity: Let $C = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $T(cA+B) = C(cA+B) = cCA+CB = cT(A)+T(B)$. Next, a simple computation shows

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

b. $[T]_\beta$ row reduces to

$$U = \begin{pmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the null space and range of $[T]_\beta$ have bases $\{(1,0,-1,0),(0,1,0,-1)\}$ and $\{(1,0,1,0),(0,1,0,1)\}$, respectively. Taking the inverse coordinate maps, we see that $\{\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\}$ and $\{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\}$ are respective bases of the null space and range of T.
c. Let $\beta' = \{A_1, A_2, A_3, A_4\}$. A simple computation shows that $T(A_i) = 0$ for $i \in \{1,2\}$ and $T(A_i = 2A_i$ for $i \in \{3,4\}$. Thus

$$[T]_{\beta'} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

d. In general we have $[T]_{\beta'} = Q^{-1}[T]_\beta Q$ where $Q = [I_V]_{\beta'}^{\beta}$. In this case

$$\begin{aligned} Q &= [I_{\mathrm{M}_{2\times 2}(\mathbb{R})}]_{\beta'}^{\beta} \\ &= ([A_1]_\beta\ [A_2]_\beta\ [A_3]_\beta [A_4]_\beta) \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}. \end{aligned}$$

2. Let $S = \{p_1, p_2, \ldots, , p_5\} \subseteq P_3(\mathbb{R})$, where $p_1 = 1 + 0x - x^2 + x^3, p_2 = 1 + x + 0x^2 + 3x^3, p_3 = 1 + 2x + x^2 + 5x^3, p_4 = 2 + 3x + x^2 + 8x^3, p_5 = 1 + x + 3x^2 + 3x^3$.
   a. (15 pts.) Let $W = \text{span}(S)$. Select a basis for W FROM AMONG THE ELEMENTS OF S.

   SOLUTION: First set $v_i = [p_i]_\beta$, where β is the standard basis, and then put the v_i as columns into a matrix to get

$$B = \begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 3 & 1 \\ -1 & 0 & 1 & 1 & 3 \\ 1 & 3 & 5 & 8 & 3 \end{pmatrix}$$

   We can row reduce B to

$$U = \begin{pmatrix} \boxed{1} & 1 & 1 & 2 & 1 \\ 0 & \boxed{1} & 2 & 3 & 1 \\ 0 & 0 & 0 & 0 & \boxed{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   Since a basis for the column space of U consists of the 1st, 2nd and 5th columns of U, the vectors v_1, v_2, v_5 form a basis for $\text{span}\{v_i\}$. Translating this back in terms of matrices, we conclude that $\{p_1, p_2, p_5\}$ is a basis for W.

   b. (5 pts.) Is $W = P_3(\mathbb{R})$? Explain.

   SOLUTION: No. $\dim W = 3$ and $\dim P_3(\mathbb{R}) = 4$.

3. Let $T : V \to W$, $U : W \to Z$ be linear. Assume V,W and Z are finite-dimensional.
   a. (10 pts.) Prove: T onto $\Rightarrow r(UT) = r(U)$.
   b. (10 pts.) Prove: U one-to one $\Rightarrow r(UT) = r(T)$. [HINT: the Dimension Theorem will help here.]

   SOLUTION:
   a. Suppose T is onto. I claim that $\text{R}(UT) = \text{R}(U)$, from which it follows immediately that $r(UT) = r(U)$. Take $z \in \text{R}(UT)$. Then there is a $v \in V$ such that $UT(v) = U(T(v)) = z$. But then $z = U(w)$, where $w = T(z)$. Thus $z \in \text{R}(U)$. Going the other way, suppose $z \in \text{R}(U)$. Then there is a $w \in W$ such that $U(w) = z$. Since T is onto, there is a $v \in V$ such that $T(v) = w$. But then $UT(v) = U(T(v)) = U(w) = z$. Thus $z \in \text{R}(UT)$.
   b. Suppose U is one-to-one. I claim that $\text{N}(UT) = \text{N}(T)$, in which case $n(UT) = n(T)$ and by the dimension theorem

$$\begin{aligned} r(UT) &= \dim V - n(UT) \\ &= \dim V - n(T) \\ &= r(T) \end{aligned}$$

   So to prove the claim, first take $v \in \text{N}(T)$. Then $UT(v) = U(T(v)) = U(0_W) = 0_Z$. Thus $v \in \text{N}(UT)$. Similarly take $v \in \text{N}(UT)$. Then since $UT(v) = U(T(v)) = 0_Z$, we see that $T(v) \in \text{N}(U)$. But then $T(v) = 0_W$ since U is one-to one. Thus $v \in \text{N}(T)$.