Math 54 — Professor K. A. Ribet

Midterm Exam — February 14, 2007

This exam was an 80-minute exam. It began at 3:40PM. There were 4 problems, for which the point counts were 7, 12, 7 and 4. The maximum possible score was 30.

> Please put away all books, calculators, and other portable electronic devices—anything with an ON/OFF switch. You may refer to a single 2-sided sheet of notes. When you answer questions, explain in words what you are doing: your paper is your ambassador when it is graded. Correct answers without appropriate supporting work will be regarded with great skepticism. Incorrect answers without appropriate supporting work will receive no partial credit. Please write your name on each page of this exam. At the conclusion, please hand in your paper to your GSI.

1. *Label the following statements as TRUE or FALSE, giving a short justification for your choice. There are six parts to this problem, two per page.*

a. *If the span of $v_1, \dots, v_n$ contains w_1, w_2 and w_3, it contains the span of these three vectors.*

This is true: see (3.33) on page 173 of the textbook.

b. *If B is an $n \times 5$ matrix, the set of matrices $A \in M_{mn}$ such that $AB = 0$ is a subspace of M_{mn}.*

This is true; you can verify closure under scalar multiplication and addition very quickly. Also, the space is non-empty–it contains the 0-matrix A.

c. *If u is a vector and v is a non-zero vector, the projection of u on v is perpendicular to u.*

This is complete nonsense. The projection is parallel to v. (It's a multiple of v, more precisely.) On the other hand, u and v probably aren't perpendicular.

d. *If v and w are column vectors of length n and m, the set of $m \times n$ matrices A satisfying $Av = w$ either has no solutions or an infinite number of solutions.*

This could be the most ridiculous question I ever made up. Since n and m are not specified, if the question is false for one set of choices of n and m, then it's simply false. Take $n = m = 1$. Then v, w and A just amount to single numbers. We are asking: when v and w are numbers, could the equation $Av = w$, where A is a number, have (for example) a single solution in A? Well, yes: if v is a non-zero number, A must be w/v. I have no idea what led me to write this question.

e. *If W and W' are subspaces of a vector space V, the set of vectors in V that belong to both W and W' is a subspace of V.*

True. Once again, having closure for W and W' gives closure for V (which is called the intersection of W and W' and written $W \cap W'$).

f. *If v_1 and v_2 are non-zero vectors of $\mathbf{R}^3$ that form an angle of $120°$, then there is a unique scalar c so that v_1 and $v_2 - c \cdot v_1$ are perpendicular.*

Sure, this is true: c is $\dfrac{v_1 \cdot v_2}{v_1 \cdot v_1}$. The dot product of v_1 with itself is non-zero because v_1 is non-zero. Note that we haven't used anything about $120°$ and don't care whether or not v_2 is non-zero.

2. *Use Gaussian elimination to determine whether or not the matrix* $\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 2 & 1 \end{bmatrix}$ *has an inverse. If it does, find the inverse.*

The inverse is $\begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$. As is typical of Gaussian elimination problems, you can get there by many paths. Just do row operations on the original matrix to get to I_3. Simultaneously, do the same row operations on I_3. The result of applying the operations to I_3 is the required inverse.

3. *Let A be the 4×4 matrix* $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 6 \\ 4 & 0 & 1 & 7 \end{bmatrix}$. *Find a basis for the null space of A. Also, find a basis for the subspace of $\mathbf{R}^4$ spanned by the columns of A.*

The null space is the set of $x \in R^4$ (thought of as vertical columns) such that $Ax = 0$. By Gaussian elimination, you can find all x. There's one free variable—all x are multiplies of a single non-zero solution. One non-zero solution to $Ax = 0$ seems to be $[-2, -3, 1, 1]$. (I did this at home, and the numbers look familiar.) This single element is a basis of the null space. Because of the relation that -2 times the first column plus -3 times the second column plus the third column is the negative of the fourth column, the span of the four columns is the span of the first three columns. The first three columns are linearly independent; indeed, there is no element of the null space, other than $[0, 0, 0, 0]$, whose last entry is 0. Hence the first three columns form a basis for the column space. In fact, any three out of the four columns form a basis for the column space. Here, I'm taking the liberty of using the shorthand "column space" for "subspace spanned by the columns." But you all knew that, right?

4. *Suppose that $x_1, \ldots, x_k$ are elements of $\mathbf{R}^n$ and that A is an $m \times n$ matrix. If the products $Ax_1, \ldots, Ax_k$ are linearly independent vectors in $\mathbf{R}^m$, show that the vectors $x_1, \ldots, x_k$ are linearly independent.*

This was a homework problem. I put it on the exam because students were asking about it all week. To show that the xs are linearly independent, start with a relation $c_1x_1 + \cdots + c_kx_k = 0$; the aim is to show that all the cs are 0. Multiply both sides of the equation by A, and rewrite the product of A and the left-hand side as $c_1(Ax_1) + \cdots + c_k(Ax_k)$. The product of A and the RHS (which is 0) is 0. Thus $c_1(Ax_1) + \cdots + c_k(Ax_k) = 0$. Since the Ax_i are linearly independent, the coefficients c_i are all 0, which is what we wanted to prove.