MATH 110-S2
SECOND IN-CLASS EXAM

Please write your name on each blue-book, and the number of blue-books used, if you use more than one.

You have until 9:30. Write all proofs in *full sentences* and show your work whenever possible. There are three problems plus and extra credit question, skip ahead if you get stuck.

Good luck!

(1) (a) (6 pts) Define the notion of Linear operator. Define the notion of null space of a linear transformation. State The Dimension Theorem.
Answer: A *linear operator* is a linear transformation for which domain and codomain coincide. The *null space* of a linear transformation is the preimage of the zero vector. The Dimension Theorem states that given $T \in \mathcal{L}(V, W)$ with V finite dimensional, then $\dim V = \dim N(T) + \dim R(T)$.

(b) (6 pts) Define the notion of standard representation of a vector space with respect to an ordered basis . Define what it means for two vector spaces to be isomorphic. Define the notion of dual basis.
Answer: Let V be a vector space over a field $\mathbb{F}$ and let $\beta = \{x_1, \dots, x_n\} \subseteq V$ be an ordered basis. The *standard representation* of V with respect to β is the linear transformation $\phi_\beta : V \to \mathbb{F}^n$ such that $\phi_\beta(x_i) = e_i$ for $i = 1, \dots, n$. Two vector spaces V, W defined over the same field $\mathbb{F}$, are *isomorphic* if there exists an invertible linear transformation $T \in (V, W)$. Let V be vector space with basis $\beta = \{x_1, x_2, \dots, x_n\}$. The dual basis β^* of β is the subset $\{f_1, f_2, \dots, f_n\}$ of V^* such that $f_i(x_j) := \delta_{ij}$.

(2) (12 pts) Label each of the following statements as True or False. Justify your answers.

(a) A function $T : \mathbb{Q}^2 \to \mathbb{Q}^2$ is a linear transformation if and only if $T(cx) = cTx$ for all $x \in \mathbb{Q}^2$ and for all $c \in \mathbb{Q}$.
Answer: FALSE. The function that is equal to the identity on the union of the coordinate axes and zero elsewhere has the desired property but is not additive because

$$(1, 0) + (0, 1) = (1, 1) \neq 0 = 0(1, 1)\,.$$

(b) If V is vector space over a field $\mathbb{F}$ then there exist integers m, n such that V is isomorphic to $M_{m\times n}(\mathbb{F})$.

Answer: FALSE. If V is infinite-dimensional, it cannot be isomorphic to a finite dimensional vector space.

(c) $T \in \mathcal{L}(\mathbb{F}^n)$ is an isomorphism if and only if there exists a $U \in \mathcal{L}(\mathbb{F}^n)$ such that $UT = I_{\mathbb{F}^n}$.
Answer: TRUE. A linear transformation between two vector spaces of the same dimension is an isomorphism if and only if it is one-to-one if and only if it has a right inverse.

(3) (26 pts) This problem is divided in four parts. For each part, explain carefully all the steps that led you to the solution. Let $\mathbb{F}$ be the field with two elements and let V be a two dimensional vector space over $\mathbb{F}$.

(a) Compute the number of ordered bases of V.
Answer: Fix a basis $\{x_1, x_2\}$ of V. The other possible bases are $\{x_1, x_1+x_2\}$ and $\{x_2, x_1 + x_2\}$. Taking the order into account, there is a total of six ordered bases.

(b) Compute the number of elements of $\mathcal{L}(V)$.
Answer: $\mathcal{L}(V)$ is a vector space of dimension 4 over a field of 2 elements and therefore consists of 16 elements.

(c) Compute the number of isomorphisms in $\mathcal{L}(V)$.
Answer: By adjunction, this is the same as the number of ordered bases as computed in part (a) i.e. 6

(d) Compute the number of operators in $\mathcal{L}(V)$ whose range is one-dimensional.
Answer: The range is a subspace and so its dimension is in between 0 and 2. Linear operators with a two-dimensional range are isomorphism and so there are 6 of them. The only linear operator with a zero-dimensional range is the zero linear operator. Thus the total number of linear operators with a one-dimensional range is $16 - 6 - 1 = 9$.

EXTRA CREDIT (5pts) Let $\mathbb{F}$ be an arbitrary field and let V be an arbitrary vector space over $\mathbb{F}$. Answer questions (a), (b), (c), (d) in Problem (3).

Answer: If either $\dim V$ or $|\mathbb{F}|$ are infinite, the answer is infinity by adjunction. for all questions. If not, $\dim V = n$ and $|\mathbb{F}| = q$. Then the number of ordered bases (also equal, by adjunction, to the number of isomorphisms in $\mathcal{L}(V)$) is

$$(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$$

and $|\mathcal{L}(V)| = q^{n^2}$. The number of operator whose range is one-dimensional is equal to the number of one-dimensional subspaces times the number of non-zero vectors in the dual space i.e.

$$\frac{(q^n - 1)^2}{q - 1}$$