MATH 54 – OLD MIDTERM #2

Problem #1 (a). Find the eigenvalues of the matrix

$$\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}.$$

(b). Compute the Wronskian of the functions

$$y_1(x) = e^x \cos x, \quad y_2(x) = e^x \sin x.$$

Problem #2. Find the determinant of the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 4 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

Problem #3. Apply the Gram–Schmidt process to convert the vectors

$$\mathbf{v}_1 = (1, 2, 1), \quad \mathbf{v}_2 = (1, -1, 1), \quad \mathbf{v}_3 = (1, 2, -1)$$

into an *orthonormal* basis of $\mathbb{R}^3$.

Problem #4. Let A be a real $n \times n$ *symmetric* matrix.

Prove that if $\mathbf{v}_1, \mathbf{v}_2$ are eigenvectors of A corresponding to the *distinct* eigenvalues λ_1, λ_2, then

$$\mathbf{v}_1, \mathbf{v}_2 \text{ are orthogonal.}$$

Problem #5. Let A be a real $n \times n$ matrix and consider the symmetric matrix $B = A^T A$. Show that if λ is an eigenvalue of B, then

$$\lambda \geq 0.$$

(Hint: Since λ is an eigenvalue, we have $B\mathbf{v} = \lambda\mathbf{v}$ for some eigenvector $\mathbf{v} \neq \mathbf{0}$. Now use the dot product.)