MATH 185 — MIDTERM 2

Problem #1. Show by a direct calculation that

$$\int_C \frac{1}{z}\, dz = 2\pi i,$$

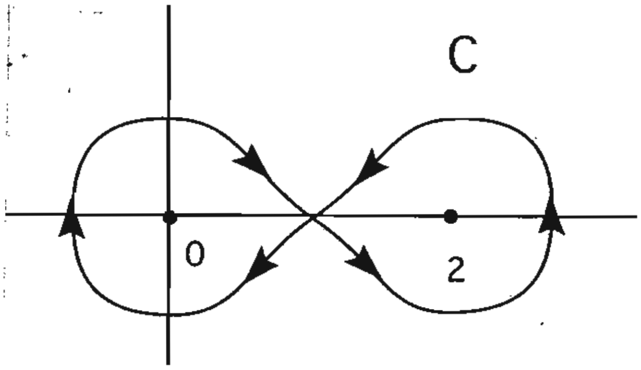
where C denotes the positively oriented unit circle, centered at 0.

(Do not use the Cauchy Integral Formula.)

Problem #2. Calculate

$$\int_C \frac{3z+1}{z(z-2)^2}\, dz$$

for the contour C as drawn:



Problem #3. Show that if f is analytic on and inside a simple closed curve C, then

$$\int_C f\, dz = 0.$$

(Hints: You may assume f is continuously differentiable. Use Green's Theorem, which says

$$\int_C P\, dx + Q\, dy = \iint_R Q_x - P_y\, dA,$$

where R is the region inside C.)

Problem #4. Assume f is analytic within the disk $|z| < R$. Show how to use *Cauchy's integral formula* to write the Taylor's series expansion

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n.$$

for $|z| < R$.

(You do not need to show rigorously that this series converges.)

Problem #5. Suppose that f is an entire function. Assume for the positive integer m that

$$|f(z)| \leq M(1 + |z|^m)$$

for some constant M and all $z \in \mathbb{C}$.

Show f is a polynomial of degree at most m. That is,

$$f(z) = a_0 + a_1 z + a_2 z^2 \cdots + a_m z^m,$$

for appropriate coefficients $a_0, a_1, \ldots, a_m$.

Problem #6. Suppose that f is analytic in the disk D given by $|z - z_0| < R$. Assume also that

$$|f(z)| \leq |f(z_0)| \qquad \text{for all } z \in D.$$

Prove that $|f|$ is constant within D.

(Hints: Do not just quote the Maximum Modulus Principle. Instead, first explain why

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\theta})\, d\theta,$$

where C denotes the circle with center z_0 and radius $0 < \rho < R$. Finish the proof from here.

In fact f is constant within D, but you do not need to show this.)