Spring 2001

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# MATH 54 FINAL (yellow)

*Do not write your answers on this sheet.* Instead please write your name, your student ID, your TA's name, your section time, "yellow," and all your answers in your blue books. **IMPORTANT:** Problems 1–16 are multiple choice or short answer questions; please write your answers to these *and nothing else* on the first page of your first blue book; for these problems you can get full credit for the answer alone. But to get partial credit for wrong answers or to get credit on problems 17–20, you must show your work on later pages in your blue book, clearly labelled by problem number. Total: 20 problems, 200 pts., 2 hours and 50 minutes.

**Math 49 students doing linear algebra only:** do only problems 1, 2, 3, 4, 6, 8, 9, 10, 11, 17. (You have 2 hours.)

**Math 49 students doing DE's only:** do only problems 5, 7, 12, 13, 14, 15, 16, 18, 19, 20. (You have 2 hours.)

**Math 49 students doing PDE's/Fourier series only:** do only problems 12, 18, 20. (You have 1 hour.)

**(1)** (5 pts.) The *first row* of the inverse of the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$ is

(A) $\begin{bmatrix} 0 & -2 & 1 \end{bmatrix}$
(B) $\begin{bmatrix} 0 & -3 & -6 \end{bmatrix}$
(C) $\begin{bmatrix} 0 & 6 & -3 \end{bmatrix}$
(D) $\begin{bmatrix} 2 & 1 & 0 \end{bmatrix}$
(E) The inverse does not exist.

**(2)** (5 pts.) The determinant of $\begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 5 & 4 \end{bmatrix}$ equals

(A) $-120$
(B) $-24$
(C) 0
(D) 24
(E) 120

**(3)** (5 pts.) For what real numbers $c$ is the matrix $\begin{bmatrix} 1 & c \\ 0 & 2 \end{bmatrix}$ diagonalizable?

(A) all real numbers $c$
(B) all nonzero real numbers $c$
(C) $c = 0$ only
(D) all real numbers except 1 and 2
(E) It is never diagonalizable.

**(4)** (6 pts.) The unique line $y = mx + b$ best approximating the data points $(1, 2)$, $(2, 3)$, $(3, 5)$, in the sense of least squares (i.e., minimizing $\sum_{i=1}^{3}(y_i - (mx_i + b))^2$, where $(x_i, y_i)$ for $i = 1, 2, 3$ are the three points) is

(A) $y = x + 3/2$.
(B) $y = x + 4/3$.
(C) $y = (3/2)x + 1/3$.
(D) $y = (3/2)x + 1/6$.
(E) The best approximating line is not uniquely determined by the given data.

In problems 5 to 8, write "TRUE" (not just T) if the statement is always true, "FALSE" if it is sometimes false. *No explanation required.*

**(5)** (6 pts.) If $A$ is a square matrix, then every solution $\mathbf{x}(t)$ to the system $\mathbf{x}' = A\mathbf{x}$ is a linear combination of the columns of $e^{tA}$.

**(6)** (6 pts.) If $A$ is a square matrix, and the characteristic polynomial of $A$ is $(x-6)^2(x-7)^3$, then there exist two linearly independent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that $A\mathbf{v}_1 = 6\mathbf{v}_1$ and $A\mathbf{v}_2 = 6\mathbf{v}_2$.

**(7)** (6 pts.) If $y_1(t)$, $y_2(t)$, $y_3(t)$ are solutions to the differential equation $y'' - ty = 0$ on $(-\infty, \infty)$ then the Wronskian $W(y_1, y_2, y_3)(t)$ is the zero function.

**(8)** (6 pts.) If $A$ and $B$ are symmetric $2 \times 2$ matrices, then $AB$ is symmetric.

In problems 9–13, write "YES" if the given set is a vector space under the usual addition and scalar multiplication, and "NO" otherwise. If "YES," give also the dimension (write "$\infty$" if it's an infinite-dimensional vector space).

**(9)** (7 pts.) The set of solutions to $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

**(10)** (7 pts.) The set of all eigenvectors of the matrix $\begin{bmatrix} 4 & 0 \\ 0 & 7 \end{bmatrix}$, including the zero vector.

**(11)** (7 pts.) The set of all singular $2 \times 2$ matrices.

**(12)** (7 pts.) The set of periodic functions $f : \mathbb{R} \to \mathbb{R}$ of period 3.

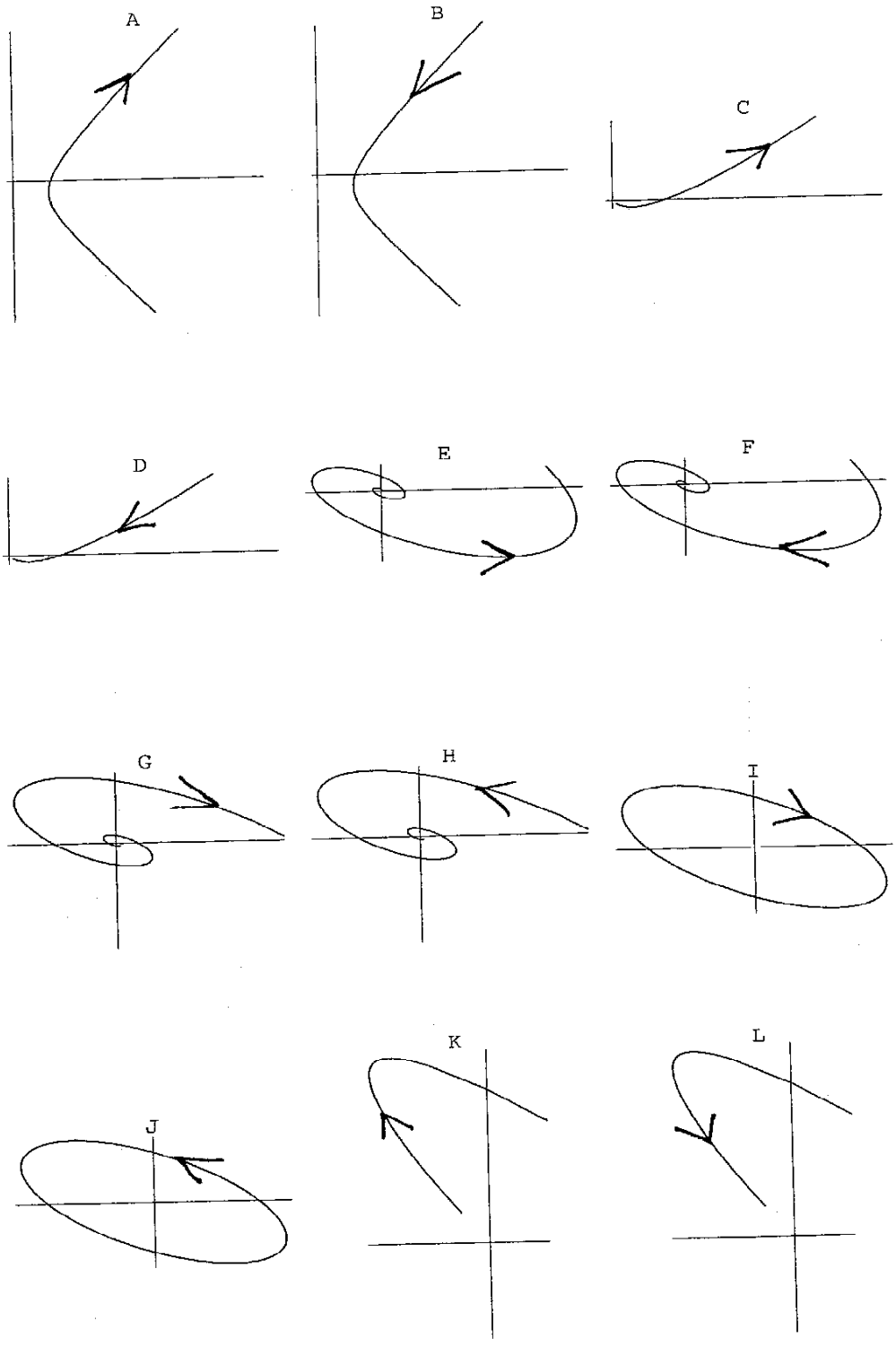
**(13)** (7 pts.) The set of functions $y(t)$ satisfying $y'' + ty' + e^t y = 0$ and $y(2) = 0$.

In problems 14–16, write the letter (A, B, ..., or L) labelling the graph on the next page that shows part of a trajectory of a solution to $\mathbf{x}' = A\mathbf{x}$.

**(14)** (10 pts.) $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

**(15)** (10 pts.) $A = \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix}$

**(16)** (10 pts.) $A = \begin{bmatrix} -26 & -60 \\ 15 & 10 \end{bmatrix}$


A
B
C
D
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L

**(17)** (25 pts.) Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of the matrix $A = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}$ such that $\begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$ is one of the vectors in the basis. (Hint: once you find your answer, it is easy to check.)

**(18)** The function $f : \mathbb{R} \to \mathbb{R}$ is an even periodic function of period 4 such that

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1 \\ 4, & \text{if } 1 \le x < 2. \end{cases}$$

**(a)** (15 pts.) Write out the first four *nonzero* terms in the Fourier series for $f(x)$, starting with the constant term.

**(b)** (5 pts.) What does the Fourier series converge to, when $x = 1$?

**(19)** (20 pts.) Find all possibilities for $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ given that $x_1(t)$ and $x_2(t)$ are functions satisfying

$$\begin{aligned} \frac{dx_1}{dt} &= x_1(t) + 4x_2(t), \\ \frac{dx_2}{dt} &= -x_1(t) + 5x_2(t), \\ x_1(0) &= 1, \\ x_2(0) &= 1. \end{aligned}$$

**(20)** (25 pts.) Find a function $u(x,t)$ defined for $0 \le x \le \pi$ and $t \ge 0$ satisfying

- $tu_{xx} = u_t$,
- $u(0,t) = 0$ for all $t \ge 0$,
- $u(\pi,t) = 0$ for all $t \ge 0$, and
- $u(x,0) = 5\sin x + 7\sin(2x)$ for $0 \le x \le \pi$.

(Warning and hint: this is not exactly the heat equation, but the same technique used to solve the heat equation will work here! You may assume without proof that given $L > 0$, all eigenvalues of the boundary value problem

$$y'' + \lambda y = 0, \qquad y(0) = y(L) = 0$$

are positive.)

This is the end! At this point, you may want to look over the exam to make sure you have not omitted any problems. (Note that problem 18 has two parts, and that problems 9–13 require you to give the dimension if the answer is "YES.")