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Math54 Midterm I Solutions

This is a closed everything exam, except a standard one-page cheat sheet (on one-side only). You need to justify every one of your answers. Completely correct answers given without justification will receive little credit. Problems are not necessarily ordered according to difficulties. You need not simplify your answers unless you are specifically asked to do so.

Problem	Maximum Score	Your Score
1	5	
2	19	
3	19	
4	19	
5	19	
6	19	
Total	100	

1. (5 Points) Write your personal information below.

   Your Name: ______________________________

   Your GSI: ______________________________

   Your SID: ______________________________

2. (19 Points) Show that you need at least m vectors to span a linear space of dimension m.

   Proof: Let $v_1, \cdots, v_k$ be a set of spanning vectors. Remove all the redundant vectors from $v_1, \cdots, v_k$. The resulting set of vectors must be linearly independent and still span the linear space, and hence must be a basis for the linear space. By definition, the number of vectors in this basis is the dimension. It follows that $m \leq k$.

3. (19 Points) Find all invertible $n \times n$ matrices A such that $A^2 + A = 0$.

   Solution: We rewrite the equation as $A(A+I) = 0$. Since A is invertible, its inverse exists. Hence $A^{-1}A(A+I) = 0$, which leads to $A + I = 0$, or $A = -I$, which is indeed invertible.

4. (19 Points) Consider a linear system of equations $A\,x = b$, where

$$A = \begin{pmatrix} 0 & k & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1+k \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

- For which values of k does the system have a unique solution and what is it?

  Solution: The argumented matrix is

$$\begin{pmatrix} 0 & k & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1+k & 2 \end{pmatrix}.$$

  Swap the first two rows and do one elementary row operation with the first and third rows, we have

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & k & 1 & 1 \\ 0 & 1 & k & 1 \end{pmatrix}.$$

  Since k might be 0, we swap the second and third rows to get

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & k & 1 \\ 0 & k & 1 & 1 \end{pmatrix}.$$

  Perform one more row elimination, we obtain

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & k & 1 \\ 0 & 0 & 1-k^2 & 1-k \end{pmatrix}. \tag{1}$$

  If $k \neq \pm 1$, we have a unique solution

$$x = \begin{pmatrix} 1 - 2/(k+1) \\ 1/(k+1) \\ 1/(k+1) \end{pmatrix}.$$

- For which values of k does the system have no solution?

  Solution: $k = -1$. This is the case where the matrix in equation (1) becomes

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

  meaning no solutions.

- For which values of k does the system have infinite number of solutions and what are they?

  Solution: $k = 1$.

  This is the case where the matrix in equation (1) becomes

  $$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

  meaning infinite number of solutions of the form

  $$x = \begin{pmatrix} 0 \\ 1-t \\ t \end{pmatrix},$$

  where t is any constant.

5. (19 Points) Let V be a subspace in $\mathcal{R}^3$. Show that there exists a 3×3 matrix A such that $V = \mathbf{im}(A)$.

   Proof: If $V = \{0\}$, then let A be the zero matrix in $\mathbf{R}^{3\times 3}$. Otherwise, let $v_1, \cdots, v_k$ be a basis for V with $k = \mathbf{dim}(V) \leq 3$. If $k = 1$, we let $A = [v_1, v_1, v_1]$. If $k = 2$, we let $A = [v_1, v_2, v_1]$, and we let $A = [v_1, v_2, v_3]$ if $k = 3$. In all cases we have $V = \mathbf{im}(A)$ by construction.

6. (19 Points) Let $\mathcal{P}$ be the set of all polynomials, i.e.,

$$\mathcal{P} = \{\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n, \text{ where } \alpha_0, \alpha_1, \cdots, \alpha_n \in \mathcal{R}, \text{ and } n \geq 0 \text{ is any integer.}\}$$

Define a linear transformation $T : \mathcal{P} \to \mathcal{P}$ as

$$T\left(f(x)\right) = x\, f(x),$$

for any $f(x) \in \mathcal{P}$.

(a) Find $\mathbf{ker}(T)$.

Solution: Let $f(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$ be a polynomial in $\mathbf{ker}(T)$. This means

$$T\left(f(x)\right) = x\, f(x) = \alpha_0 x + \alpha_1 x^2 + \cdots + \alpha_n x^{n+1} = 0.$$

It follows that $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 0$. Hence $f(x) = 0$. $\mathbf{ker}(T)$ contains only the zero polynomial.

(b) Find a polynomial $g(x)$ that is in $\mathcal{P}$ but not in $\mathbf{im}(T)$.

Solution: Let $f(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$ be a polynomial in $\mathbf{im}(T)$. This implies that there is a polynomial $h(x) = \beta_0 + \beta_1 x + \cdots + \beta_n x^n$ in $\mathcal{P}$ such that $f(x) = T(h(x)) = xh(x) = \beta_0 x + \beta_1 x^2 + \cdots + \beta_n x^{n+1}$. Hence the constant term in $f(x)$ is zero. Conversely, any polynomial with zero constant term is in $\mathbf{im}(T)$. Hence the constant polynomial $g(x) = 1$ must not be in $\mathbf{im}(T)$.

(c) Find a basis for $\mathbf{im}(T)$.

Solution: As argued before, $\mathbf{im}(T)$ consists of all polynomials with zero constant term, a basis for $\mathbf{im}(T)$ is $\{x, x^2, \cdots, x^n, \cdots\}$.