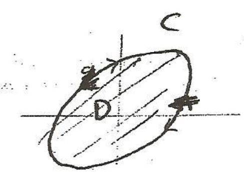
Midterm #3, Rezakhanlou Math 53 FA 06

1. Let C be the parametric curve

$$x = 3\cos\theta + 2\sin\theta, \quad y = 3\cos\theta - 2\sin\theta, \quad 0 \le \theta \le 2\pi.$$



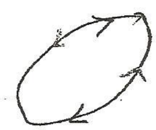
(a) *(7 pts)* Find the area inside C.

$$\text{Area of } D = \frac{1}{2}\int_{-C} x\,dy - y\,dx \quad \leftarrow \text{to get correct sign}$$

$$= \frac{1}{2}(-1)\int_0^{2\pi} d\theta \left[(3\cos\theta + 2\sin\theta)(-3\sin\theta - 2\cos\theta) - (3\cos\theta - 2\sin\theta)(-3\sin\theta + 2\cos\theta)\right]$$

$$= \frac{1}{2}\int_0^{2\pi} 12\, d\theta = 12\pi.$$

(b) *(9 pts)* Find the work done by the vector field $\mathbf{F}(x, y) = (3x^2 + 4y)\mathbf{i} + (8x + 9y^4)\mathbf{j}$ on a particle which moves clockwise along C.



Since D is on the right as we move clockwise,

$$\text{Work} = \int_C \vec{F} \cdot d\vec{s} = \int_C P\,dx + Q\,dy$$

$$= -\iint_D \left\{\frac{\partial}{\partial x}[8x + 9y^4] - \frac{\partial}{\partial y}[3x^2 + 4y]\right\} dx\,dy$$

$$= -4 \text{ Area of } D = -48\pi.$$

2. Let $\mathbf{F} = -e^{y+z^2}\sin(x+z)\mathbf{i} + (1+e^{y+z^2}\cos(x+z))\mathbf{j} + e^{y+z^2}(2z\cos(x+z) - \sin(x+z))\mathbf{k}$.

(a) *(10 pts)*
Find a function f such that $\mathbf{F} = \nabla f$.

$$f_x = e^{-y+z^2}\sin(x+z) \Rightarrow f = e^{y+z^2}\cos(x+z) + g(y,z).$$

$$f_y = e^{y+z^2}\cos(x+z) + 1 \Rightarrow g_y = 1, \quad g = y + h(z)$$

From $f_z = e^{y+z^2}(2z\cos(x+z) - \sin(x+z)) \Rightarrow h_z = 0$.

Thus $f = e^{y+z^2}\cos(x+z) + y + \text{Const}$.

(b) *(5 pts)*
Is there a vector field $\mathbf{G}$ such that $\mathbf{F} = \nabla \times \mathbf{G}$? If such G exists,

then $\operatorname{div} F = 0$. But

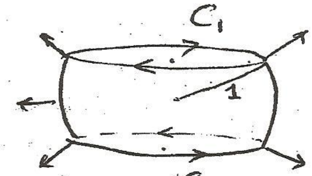
$$\operatorname{div} F = -e^{y+z^2}\cos(x+z) + e^{y+z^2}\cos(x+z)$$

$$+ e^{y+z^2}\Big[2z\,(2z\cos(x+z) - \sin(x+z))$$

$$+ 2\cos(x+z) - 2z\sin(x+z) - \cos(x+z)\Big]$$

$$= e^{y+z^2}\left[4z^2\cos(x+z) - 4z\sin(x+z)\right] \neq 0$$

3. *(17 pts)*
Let S denote the portion of the sphere $x^2 + y^2 + z^2 = 1$ with $\frac{1}{2} \le z \le \frac{1}{2}$, oriented outward. Draw a figure including the orientation of each boundary curve. State Stokes' theorem for S and evaluate $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = y\mathbf{i} - 2xz\mathbf{j} + ye^z\mathbf{k}$.



$$\iint_S \nabla \times F \cdot dS = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

where C_1, C_2 are the boundary curves

C_1 : $\vec{r}_1(t) = \langle \frac{\sqrt{3}}{2} \cos t, -\frac{\sqrt{3}}{2} \sin t, \frac{1}{2} \rangle$

C_2 : $\vec{r}_2(t) = \langle \frac{\sqrt{3}}{2} \cos t, \frac{\sqrt{3}}{2} \sin t, -\frac{1}{2} \rangle$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle -\frac{\sqrt{3}}{2} \sin t, -\frac{\sqrt{3}}{2} \cos t, ye^z \rangle \cdot \langle -\frac{\sqrt{3}}{2} \sin t, -\frac{\sqrt{3}}{2} \sin t, 0 \rangle \, dt$$

$$= \frac{3}{4} 2\pi = \frac{3\pi}{2}$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle \frac{\sqrt{3}}{2} \sin t, \frac{\sqrt{3}}{2} \cos t, ye^z \rangle \cdot \langle -\frac{\sqrt{3}}{2} \sin t, \frac{\sqrt{3}}{2} \cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} 3/4 \, (\cos^2 t - \sin^2 t) = \int_0^{2\pi} 3/2 \, \cos 2t \, dt = 0 .$$

Thus $\iint_S \nabla \times F \cdot dS = 3/2 \, \pi$

4. *(13 pts)* Use the change of variables $x = r\cos\theta$, $y = \frac{r}{\sqrt{2}}\sin\theta$ to evaluate

$$\iint_D \cos(2x^2 + 4y^2)\,dx\,dy,$$

where D is the region in the first quadrant bounded by the ellipse $x^2 + 2y^2 \leq 1$.

$x = r\cos\theta$, $y = \frac{r}{\sqrt{2}}\sin\theta$, $D = \{(r,\theta) : 0 \leq r \leq 1,\ 0 \leq \theta \leq \frac{\pi}{2}\}$

$x_r = \cos\theta$ $\quad y_r = \frac{1}{\sqrt{2}}\sin\theta$

$x_\theta = -r\sin\theta$ $\quad y_\theta = \frac{r}{\sqrt{2}}\cos\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & \frac{1}{\sqrt{2}}\sin\theta \\ -r\sin\theta & \frac{1}{\sqrt{2}}\cos\theta \end{vmatrix} = \frac{r}{\sqrt{2}}$$

Hence

$$\iint_D \cos(2x^2+4y^2)\,dx\,dy = \int_0^1 \int_0^{\pi/2} \cos(2r^2)\frac{r}{\sqrt{2}}\,dr\,d\theta$$

$$= \frac{\pi}{8\sqrt{2}} (\sin 2)$$

5. A surface S is given by the parametric equation $\mathbf{r}(\alpha,\theta) = \alpha\cos\theta\mathbf{i} + \alpha\sin\theta\mathbf{j} + \theta\mathbf{k}$, $0 \le \alpha \le 1$, $0 \le \theta \le \frac{\pi}{2}$.

(a) *(6 pts)* Find the tangent plane to the surface at the point $(x, y, z) = (1, 1, \pi/4)$.

$r_\theta = \langle -\alpha\sin\theta, \alpha\cos\theta, 1\rangle$, $\vec{N} = \vec{r}_\alpha \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 0 \\ -\alpha\sin\theta & \alpha\cos\theta & 1 \end{vmatrix}$

$r_\alpha = \langle \cos\theta, \sin\theta, 0\rangle$

$= \langle \sin\theta, -\cos\theta, \alpha\rangle$

Tangent plane at $(\theta = \pi/4, \alpha = \sqrt{2})$ is $\frac{\sqrt{2}}{2}(x-1) - \frac{\sqrt{2}}{2}(y-1) + \sqrt{2}(z - \frac{\pi}{4}) = 0$

(b) *(6 pts)*
Calculate $\iint_S \sqrt{x^2+y^2}\, dS$.

$$\int_0^1 \int_0^{\pi/2} \alpha\, |r_\alpha \times r_\theta|\, d\alpha\, d\theta = \int_0^1 \int_0^{\pi/2} \alpha\sqrt{1+\alpha^2}\, d\alpha$$

$$= \frac{\pi}{2}\left[\frac{(1+\alpha^2)^{3/2}}{3}\right]_0^1 = \frac{\pi}{2} \cdot \frac{2^{3/2}-1}{3}$$

(c) *(6 pts)*
Calculate $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = \langle zx, zy, xy\rangle$.

$$\iint \vec{F}\cdot d\vec{S} = \int_0^1 \int_0^{\pi/2} \langle \alpha\theta\cos\theta, \alpha\theta\sin\theta, \alpha^2\sin\theta\cos\theta\rangle \cdot \langle \sin\theta, -\cos\theta, \alpha\rangle\, d\theta\, d\alpha$$

$$= \int_0^1 \alpha^3\, d\alpha \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta$$

$$= \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{8}$$