**1**. Solve the differential equation

$$xy'' + (1-x)\,y' + y = 0$$

*Hint:* one solution is polynomial.

Laguerre: $xy'' + (1-x)y' + ny = 0$

$n = 1$

1st soln: $L_1(x) = 1 - x$ ✓

$\frac{7}{10}$

Fuch's conditions satisfied

2nd soln $= (1-x)\ln x + v$

└ Frob. series

$y' = \frac{1}{x} - \ln x - 1 + v'$

$y'' = -\frac{1}{x^2} - \frac{1}{x} + v''$

$x\left[v'' - \frac{1}{x^2} - \frac{1}{x}\right] + (1-x)\left[\frac{1}{x} - \cancel{\ln x} - 1 + v'\right] + \cancel{(1-x)\ln x} + v = 0$

$xv'' + (1-x)v' + v - \cancel{\frac{1}{x}} - 1 + \cancel{\frac{1}{x}} - 1 - 1 + x = 0$

$xv'' + (1-x)v' + v = 3 - x$ solve?

? $v = (1-x) - 3 + x = -2$ ?

$\boxed{(1-x)\ln x - 2}$

**2.** (a) Using the generating function

$$\Phi(x,h) = e^{2xh-h^2} = \sum_{n=0}^{\infty} H_n(x)\frac{h^n}{n!},$$

prove the recursion relation for the Hermite polynomials

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$$

(b) Evaluate $H_n(0)$ for all $n$.

a) $e^{2xh-h^2} = 1 + (2xh-h^2) + \frac{(2xh-h^2)^2}{2!} + \frac{(2xh-h^2)^3}{3!} + \dots$

$H_0 = 1 \quad H_1 = 2x \;, \; H_2 = 4x^2 - 2$

$H_2(x) = 2x\, H_1(x) - 2\cancel{n} H_0(x)$

$4x^2 - 2 = 2x \cdot 2x - 2(1)$

$4x^2 - 2 = 4x^2 - 2$ ✓

for all n?

b) $\Phi(0,h) = 1 - h^2 + \frac{h^4}{2!} - \frac{h^6}{3!} + \dots$

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$$\boxed{H_n(0) = \begin{cases} 0, & n \text{ odd} \\ (-1)^{n/2}\frac{n!}{(n/2)!}, & n \text{ even} \end{cases}}$$

why?

Where is you work?
Does not look a correct answer?

**3**. A string of length 3 has the zero initial velocity; the initial displacement is given by the function

$$f(x) = \begin{cases} 0.1x & \text{if } 0 \le x \le 1, \\ 0.15 - 0.05x & \text{if } 1 \le x \le 3. \end{cases}$$

Assuming the wave velocity $v = 1$, find the position of the midpoint of the string at $t = 1, 2, 3$.

$$y = \sin kx \cos kvt \qquad 3k = n\pi \qquad k = \frac{n\pi}{3}$$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{3} \cos\frac{n\pi t}{3}$$

$$y(x,0) = \sum b_n \sin\frac{n\pi x}{3} = f(x)$$

$$b_n = \frac{2}{3}\left[\int_0^1 .1x \sin\frac{n\pi x}{3}\,dx + \int_1^3 (.15 - .05x)\sin\frac{n\pi x}{3}\,dx\right]$$

$$\frac{2}{3} \cdot .1\left[\frac{-3}{n\pi} x \cos\frac{n\pi x}{3}\Big|_0^1 + \frac{9}{n^2\pi^2}\sin\frac{n\pi x}{3}\Big|_0^1\right] + \frac{2}{3}\times .15\left[\frac{3}{n\pi}\cos\frac{n\pi x}{3}\Big|_1^3 - \frac{2}{3}.05\left[\frac{3}{n\pi}x \cos\frac{n\pi x}{3}\Big|_1\right.\right.$$

$$\frac{2}{3}\times .1\left[-\frac{3}{n\pi}\cos\frac{n\pi}{3} + \frac{9}{n^2\pi^2}\sin\frac{n\pi}{3}\right] +$$

$$y = f(x+vt) + g(x-vt)$$

$\frac{4}{10}$

$$Y_{t=0} = 0 = \cancel{v} f'(x) - \cancel{v} g'(x) \qquad f'(x) = g'(x)$$

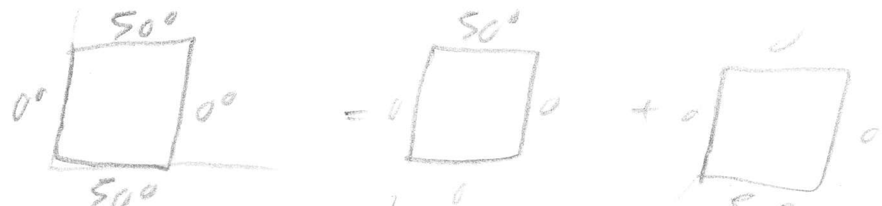
$$f(x) = g(x)$$

$$\boxed{y(1.5, 3) = 0} \qquad y = f(x+t) + f(x-t)$$

$y(1.5, 2) = .015?$ $\quad .075$ at $t=0$ $\quad$ and ...?

$y(1.5, 1) = .05$ ?

**4.** Find the steady-state temperature in a square plate of size 10 by 10 if the top and bottom sides are kept at temperature 50°, and right and left sides are held at 0°.




$\nabla^2 T = 0$

$T_1 = \sin kx \sinh ky \qquad k = \frac{n\pi}{10}$

$T_1 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{10} x \sinh \frac{n\pi y}{10}$

$T_{1\,y=10} = \sum b_n \sinh(n\pi) \sin \frac{n\pi}{10} x = 50$

$B_n = b_n \sinh(n\pi)$

$B_n = \begin{cases} \frac{200}{n\pi}, & \text{odd } n \\ 0, & \text{even } n \end{cases}$

$T_1 = \sum_{n=1}^{\infty} \frac{200}{n\pi \sinh n\pi} \sin \frac{n\pi}{10} x \sinh \frac{n\pi y}{10}$

$T_2 = \sin kx \sinh k(10-y)$

$T_2 = \sum b_n \sin \frac{n\pi}{10} x \sinh \frac{n\pi}{10}(10-y)$

$T_{2\,y=0} = \sum b_n \sinh n\pi \sin \frac{n\pi}{10} x = 50$

$b_n = \frac{200}{n\pi \sinh n\pi}$

$T_2 = \sum \frac{200}{n\pi \sinh n\pi} \sin \frac{n\pi}{10} x \sinh \frac{n\pi(10-y)}{10}$

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$$\boxed{T(x,y) = \sum_{n=1}^{\infty} \frac{200}{n\pi \sinh n\pi} \sin \frac{n\pi}{10} x \left[ \sinh \frac{n\pi y}{10} + \sinh \frac{n\pi(10-y)}{10} \right]}$$