Math 110 Spring 2008

Midterm 2

Write your name and SID on the front of your blue book. All answers and work should also be written in your blue book. You must ***JUSTIFY*** *your answers, so show your work. Partial credit will be awarded even if answers are incorrect. No notes, books, or calculators. Good luck!*

1. (20 pts.) Let $A = \begin{pmatrix} 1 & 3 & 3 \\ 6 & 4 & 6 \\ -3 & -3 & -5 \end{pmatrix}$. The characteristic polynomial of A is $f(t) = -(t^3 - 12t - 16)$.
   a.(5 pts.) Factor $f(t)$.

   SOLUTION: $f(t) = -(t-4)(t+2)^2$.

   b.(15 pts.) Find an invertible matrix Q and a diagonal matrix D such that $D = Q^{-1}AQ$. You do not need to compute Q^{-1}.

   SOLUTION: One possible pair, (Q, D), is $Q = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

2. (20 pts.)
   a.(7 pts.) Suppose the characteristic polynomial of A is $f(t) = t^4 - 1$. Use the Cayley-Hamilton Theorem to express $A^{10} + A^8$ as a linear combination of I, A, A^2, A^3.

   SOLUTION: BY the C-H Theorem, $A^4 = I$. Then $A^{10} = (A^4)^2 A^2 = IA^2 = A^2$ and $A^8 = I$. Thus $A^{10} + A^8 = A^2 + I$. □

   b.(6 pts.) Suppose T has characteristic polynomial $f(t) = (-1)^n[t^{n-2}(t-\lambda_2)(t-\lambda_3)]$, where $\lambda_2 \neq \lambda_3$ and $\lambda_i \neq 0$. Suppose further that $\dim \mathrm{N}(T) = n-2$. Is T diagonalizable? Justify your answer.

   SOLUTION: Since $\mathrm{N}(T) = E_0$ we have $\dim E_0 = n-2$ Since λ_i is an eigenvalue of multiplicity one for $i \in \{2,3\}$, we must have $\dim E_{\lambda_2} = \dim E_{\lambda_3} = 1$. This means that $\dim E_0 + \dim E_{\lambda_2} + \dim E_{\lambda_3} = (n-2) + 1 + 1 = n$. Thus T is diagonalizable. □

   c.(7 pts.) Suppose $A \in \mathrm{M}_{n\times n}(\mathbb{C})$ satisfies $A^k = 0_{n\times n}$ for some $k \geq 0$. Show that A has exactly one eigenvalue. What is it? Prove that in fact $A^n = 0_{n\times n}$. (Note that the original k may have been larger than n.)
   SOLUTION: Since A is a complex matrix, it is guaranteed to have at least one eigenvalue. Suppose $Av = \lambda v$ with $v \neq 0$. Then $A^k v = \lambda^k v$, as we proved in a hw exercise. But $A^k = 0_n$. Thus $0 = \lambda^k v$. Since $v \neq 0$, we must have $\lambda = 0$. As A only has 0 as an eigenvalue, its characteristic polynomial is $(-1)t^n$. It now follows from the C-H Theorem that $(-1)^n A^n = A^n = 0_n$. □

3. (20 pts.) Let V be an n-dimensional space and suppose $W_1, W_2 \subseteq V$ are two subspaces such that $n = \dim W_1 + \dim W_2$.
   a.(10 pts.) Prove that $V = W_1 \oplus W_2$ if and only if $W_1 \cap W_2 = \{0\}$.

   SOLUTION: The $(\Rightarrow)$ direction follows simply from the definition of direct sum. Suppose $W_1 \cap W_2 = \{0\}$. Let $\beta_1 = \{v_1, \ldots, v_k\}$ and $\beta_2 = \{v'_1, \ldots, v'_{n-k}\}$ be bases for W_1 and W_2

respectively. Note that my indexing uses the fact that $\sum \dim W_i = n$. I claim $\beta = \beta_1 \cup \beta_2$ is a basis, whence the result follows from Theorem 5.10. Since β contains n elements, it is enough to show it is linearly independent. Suppose $\sum_{i=1}^{k} a_i v_i + \sum_{j=1}^{n-k} b_j v_j' = 0$. Let $w = \sum a_i v_i \in W_1$ and $w' = \sum b_j v_j \in W_2$. Then $w_1 + w_2 = 0$ implies that $w_1 = -w_2 \in W_2$. But then $w_1 \in W_1 \cap W_2 = \{0\}$. Thus $w_1 = w_2 = 0$. Since the β_i are bases, we must have $a_i = b_j = 0$ for all i, j. □

b.(10 pts.) Now consider $T \in \mathcal{L}(V)$. Recall that $\mathrm{N}(T)$ and $\mathrm{R}(T)$ are T-invariant subspaces. Prove that $V = \mathrm{N}(T) \oplus \mathrm{R}(T)$ if and only if $T_{\mathrm{R}(T)}$, the restriction of T to $\mathrm{R}(T)$, is invertible.

SOLUTION: Note first that $\dim \mathrm{N}(T) + \dim \mathrm{R}(T) = n$ by the Dimension Theorem. Thus by part a., $V = \mathrm{N}(T) \oplus \mathrm{R}(T)$ if and only if $\mathrm{N}(T) \cap \mathrm{R}(T) = \{0\}$. This is true if and only if given $v \in \mathrm{R}(T)$, $T(v) = 0 \Rightarrow v = 0$ if and only if $T_{\mathrm{R}(T)}$ is one-to-one if and only $T_{\mathrm{R}(T)}$ is invertible. □