Math 54. Sample Answers to Second Midterm

As before, there were two versions of the midterm, distinguishable by the fact that one version gave the starting time as 9:30, and the other as 9:40. The answers given here refer to the "9:30" version. The answer for the "9:40" version is briefly given if necessary, but without details, since they are similar to the details for the answers given here.

1. (12 points) A chemist solves a nonhomogeneous system of seven linear equations in ten unknowns and finds that four of the unknowns are free variables. Can the chemist be certain that, if the right-hand sides of the equations are changed, the new nonhomogeneous linear system will have a solution? Explain.

Let A be the coefficient matrix of the linear system. It is a 7×10 matrix, and we are given that $\dim \operatorname{Nul} A = 4$. By the Rank Theorem, which says that

$$\operatorname{rank} A + \dim \operatorname{Nul} A = 10 ,$$

it follows that $\operatorname{rank} A = 6$. This is the dimension of $\operatorname{Col} A$. Since $\operatorname{Col} A$ is not all of $\mathbb{R}^7$ (where $\vec{b}$ lives), the system $A\vec{x} = \vec{b}$ is not consistent for all $\vec{b}$ (Theorem 4 on page 43).

2. (21 points) The sets

$$\mathcal{B} = \left\{ \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \right\} \qquad \text{and} \qquad \mathcal{C} = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

are bases of a vector subspace V of $\mathbb{R}^3$.

(a). Find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the coordinate vectors $[\vec{b}_1]_{\mathcal{C}}$ and $[\vec{b}_2]_{\mathcal{C}}$, where $\vec{b}_1$ and $\vec{b}_2$ are the vectors in $\mathcal{B}$. To find $[\vec{b}_1]_{\mathcal{C}}$, solve for the weights in

$$x_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}$$

by row reducing the augmented matrix of the linear system:

$$\begin{bmatrix} 3 & 1 & 5 \\ 2 & -1 & 0 \\ 1 & 2 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 5 \\ 2 & -1 & 0 \\ 3 & 1 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 5 \\ 0 & -5 & -10 \\ 0 & -5 & -10 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

so $[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Similarly, to find $[\vec{b}_2]_{\mathcal{C}}$, solve for the weights in

$$x_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

by row reducing the augmented matrix

$$\begin{bmatrix} 3 & 1 & 4 \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -5 & -5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so $[\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Therefore $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.

(b). Find $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$.

We have

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = -\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

(by the formula for the inverse of a 2×2 matrix).

(c). If $T\colon V \to V$ is a linear transformation whose $\mathcal{B}$-matrix is $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then find $[T]_{\mathcal{C}}$.

For this part you may leave the answer as a product of matrices and inverses of matrices (i.e., you do not need to carry out any matrix multiplications or inverses).

We want a matrix M satisfying $[T(\vec{x})]_{\mathcal{C}} = M[\vec{x}]_{\mathcal{C}}$ for all $\vec{x} \in V$. We have

$$[T(\vec{x})]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[T(\vec{x})]_{\mathcal{B}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[T]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[T]_{\mathcal{B}}\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}[\vec{x}]_{\mathcal{C}},$$

and therefore the answer is

$$M = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[T]_{\mathcal{B}}\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}.$$

3. (15 points) For each of the following matrices, either show that it can be diagonalized, or that it can't be diagonalized.

(a). $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

This is an upper triangular matrix, so its eigenvalues are the diagonal entries: $1, 2, 3$. Since they are distinct, the matrix is diagonalizable.

(b). $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

Again, since the matrix (call it A) is lower triangular, the eigenvalues are the diagonal entries. However, now the eigenvalues are all the same: $\lambda = 1$. The eigenspace is the null space of $A - \lambda I = A - I = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix}$. This eigenspace has dimension 1. (The matrix has two linearly independent (nonzero) columns, so its rank is 2. By the rank theorem, $\dim \operatorname{Nul} A = 1$.) Since there are not enough linearly independent eigenvectors to form a basis for $\mathbb{R}^3$, the matrix is not diagonalizable.

(For the 9:40 exam, the matrices were the transposes of the matrices here, and were interchanged.)

4. (15 points) A 5×5 matrix A has characteristic polynomial $-\lambda^3(\lambda - 1)(\lambda - 3)$.

(a). What values can $\dim \operatorname{Nul} A$ have?

The null space of the matrix is the eigenspace for the eigenvalue $\lambda = 0$. Since this is a triple eigenvalue, the null space can have dimension 1, 2, or 3.

(b). For each value n you gave in part (a), answer the following question:
If A has the above characteristic polynomial, and if $\dim \operatorname{Nul} A = n$, then is A always diagonalizable, never diagonalizable, or sometimes diagonalizable (depending on the particular matrix A)? Explain.

If $n = 1$ or $n = 2$, the matrix is not diagonalizable, since there would be at most $n + 2 < 5$ linearly independent eigenvectors. (The eigenspaces for $\lambda = 1$ and $\lambda = 3$ always have dimension 1, since these are not multiple eigenvalues.)
If $n = 3$ then the matrix is always diagonalizable, since then there would be five linearly independent eigenvectors.

5. (22 points) Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 9 \\ 5 \\ 7 \end{bmatrix}$.

(a). Let $W = \operatorname{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Use the Gram-Schmidt process to find an orthogonal basis for W.

An orthogonal basis is $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$, where

$$\vec{w}_1 = \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} ;$$

$$\vec{w}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 3 \end{bmatrix} - \frac{15}{15}\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} ;$$

$$\vec{w}_3 = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 - \frac{\vec{v}_3 \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2}\vec{w}_2 = \begin{bmatrix} 1 \\ 9 \\ 5 \\ 7 \end{bmatrix} - \frac{45}{15}\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} - \frac{6}{6}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -1 \\ 2 \end{bmatrix} .$$

(b). Let $V = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$. Find the vector in V closest to $\vec{v}_3$.

This is

$$\operatorname{proj}_V \vec{v}_3 = \frac{\vec{v}_3 \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\vec{w}_1 + \frac{\vec{v}_3 \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2}\vec{w}_2 = \frac{45}{15}\begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix} + \frac{6}{6}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 6 \\ 5 \end{bmatrix} .$$

(You may also recognize this vector as $\vec{v}_3 - \vec{w}_3$.)

(c). Find the distance between V and $\vec{v}_3$.

This distance is the distance

$$\|\operatorname{proj}_V \vec{v}_3 - \vec{v}_3\| = \left\| \begin{bmatrix} 4 \\ 8 \\ 6 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 9 \\ 5 \\ 7 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 3 \\ -1 \\ 1 \\ -2 \end{bmatrix} \right\| = \sqrt{9+1+1+4} = \sqrt{15} .$$

(Again, it is no coincidence that this is the length of $-\vec{w}_3$.)

6. (15 points) Use methods from Math 54 to find an upper bound for the integral

$$\int_0^{\pi/2} \sqrt{x \sin x}\, dx \ .$$

Your answer may be an algebraic formula involving π and square roots, but not involving integrals, limits, or infinite sums.

[**Hint:** You may recall facts about integrals from homework problems and examples in the book.]

The formula

$$\langle f, g \rangle = \int_0^{\pi/2} f(x) g(x)\, dx$$

defines an inner product on the vector space $C[0, \pi/2]$ of continuous functions on the closed interval $[0, \pi/2]$ (see Example 7 or Exercises 21 and 23 in Section 6.7). The integral in question can then be expressed in terms of this inner product when $f(x) = \sqrt{x}$ and $g(x) = \sqrt{\sin x}$. By applying the Cauchy-Schwarz inequality $|\langle f, g \rangle| \le \|f\| \|g\|$, we have

$$\int_0^{\pi/2} \sqrt{x \sin x}\, dx = \langle \sqrt{x}, \sqrt{\sin x} \rangle \le \|\sqrt{x}\| \|\sqrt{\sin x}\| \ .$$

We have

$$\|\sqrt{x}\| = \sqrt{\int_0^{\pi/2} \sqrt{x} \cdot \sqrt{x}\, dx} = \sqrt{\int_0^{\pi/2} x\, dx} = \sqrt{\frac{x^2}{2} \Big|_0^{\pi/2}} = \sqrt{\frac{\pi^2/4}{2}} = \frac{\pi}{\sqrt{8}}$$

and similarly

$$\|\sqrt{\sin x}\| = \sqrt{\int_0^{\pi/2} \sin x\, dx} = \sqrt{-\cos x \Big|_0^{\pi/2}} = \sqrt{1} = 1 \ .$$

Therefore

$$\int_0^{\pi/2} \sqrt{x \sin x}\, dx \le \frac{\pi}{\sqrt{8}} \cdot 1 = \frac{\pi}{\sqrt{8}} \ .$$

For the "9:40" exam, the answer came to $\dfrac{\pi}{\sqrt{2}} \cdot \sqrt{2} = \pi$.