Math 110 — Professor K. A. Ribet

Midterm Exam — April 5, 2005

This exam was an 80-minute exam. It began at 12:40PM. There were 4 problems, for which the point counts were 18, 10, 10 and 10. The maximum possible score was 48.

> *Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Explain your answers in full English sentences as is customary and appropriate. Your paper is your ambassador when it is graded. At the conclusion of the exam, please hand in your paper to your GSI.*

1. *Label the following statements as TRUE or FALSE, giving a short explanation (e.g., a proof or counterexample). There are six parts to this problem, which continues on page 3.*

a. *A matrix over a field is invertible if and only if it is a product of elementary matrices.*

This is TRUE. We proved the statement in class, so I won't say more.

b. *An $n \times n$ matrix with real entries is diagonalizable when considered as an element of $M_n(\mathbf{C})$ if and only if it is diagonalizable when considered as an element of $M_n(\mathbf{R})$.*

This is simply not true. We could consider, for example, the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ over $\mathbf{R}$. Its characteristic polynomial is $t^2 + 1$. Over $\mathbf{R}$, there are no eigenvalues because there are no square roots of -1. Over $\mathbf{C}$, the polynomial splits into distinct factors, so the matrix is diagonalizable. It's similar to the diagonal matrix with i and $-i$ on the diagonal.

c. *If a matrix over F has m rows and n columns, the row rank of the matrix is at most n.*

This is certainly TRUE because the row rank is the column rank; the column rank is at most the number of columns because it's the dimension of the space spanned by the columns.

d. *When A is a square matrix over F, the system of linear equations $Ax = b$ has exactly one solution if and only if the corresponding homogeneous system $Ax = 0$ has exactly one solution.*

This is again TRUE. The equation $Ax = b$ has exactly one solution precisely when b is in the range of L_A and the nullity of L_A is 0. Because A is a square matrix, L_A has nullity 0 if and only if it is onto and if and only if it is invertible. Hence $Ax = b$ has exactly one solution precisely when the nullity of L_A is 0. (The question of whether or not b is in the range then becomes moot.) Meanwhile, the equation $Ax = 0$ has exactly one solution if and only if the nullity of L_A is 0.

e. *An upper-triangular matrix with distinct diagonal entries is diagonalizable.*

Let A be the matrix. The statement is TRUE because the characteristic polynomial of A is the product $(t - a_1) \cdots (t - a_n)$, where the a_n are the numbers on the diagonal. Since the characteristic polynomial splits completely and has roots with multiplity 1, A will be diagonalizable.

f. *If A is a $n \times n$ real matrix for which $0 = (A - 2I_n)(A - 3I_n)(A - 4I_n)(A - 5I_n)$, then at least one of the numbers 2, 3, 4, 5 is an eigenvalue of A.*

TRUE again. If the numbers $2, \ldots, 5$ are not eigenvalues of A, then each of the factors $A - 2I_n$, $\ldots$, $A - 5I_n$ is invertible. Accordingly, the product $(A - 2I_n)(A - 3I_n)(A - 4I_n)(A - 5I_n)$ is invertible, which means that it is not 0. (We understand that n is positive because we never consider 0×0 matrices in the course.)

2. *Use mathematical induction to establish a formula for the determinant of an $n \times n$ matrix* $\begin{pmatrix} 0 & 0 & \cdots & 0 & c_1 \\ 0 & 0 & \cdots & c_2 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & c_{n-1} & \cdots & 0 & 0 \\ c_n & 0 & \cdots & 0 & 0 \end{pmatrix}$ *that has zero entries except possibly for the last entry of the first row, the second-to-last entry of the second row,..., the second entry of the second-to-last row, and the first entry of the last row. (Such a matrix might be called "anti-diagonal.")*

Let D_n be the indicated determinant when the size of the matrix is n. We have $D_1 = c_1$, $D_2 = -c_1c_2$, and "so on." Expanding the determinant along the first column (or last row), we get that $D_n = (-1)^{n+1} c_n D_{n-1}$. Thus D_n is

$\pm$ the product $c_1 \cdots c_n$, and the question is to see what the sign is. The sign changes when n is even, so you get something like $+$, $-$, $-$, $+$, $+$,.... The nth sign is different from the previous sign if and only if n is even. As far as I'm concerned, it's enough that you see this; I'm not super-worried about your writing $D_n = (-1)^{s_n} c_1 \cdots c_n$ with an explicit formula for s_n. To get an explicit formula, you can observe that s_n is $2 + \cdots + (n+1)$, or $\dfrac{n(n+3)}{2}$.

3. *Suppose that $A \in M_{n\times n}(F)$ is an invertible matrix over a field F. Show that there are $c_0, \ldots, c_{n-1}$ in F such that*

$$A^{-1} = c_{n-1}A^{n-1} + c_{n-2}A^{n-2} + \cdots + c_1 A + c_0 I_n.$$

(Example: if $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, *then $A^{-1} = A + I_3$.)*

By the Cayley–Hamilton theorem, $f(A) = 0$, where $f(t)$ is the characteristic polynomial of A. The polynomial $(-1)^n f(t)$ starts out $t^n + \cdots$; it looks like $t^n + a_{n-1}t^{n-1} + \cdots + a_0$. Note that a_0 is $\pm \det A$, since $f(0) = \det(A - 0 \cdot I_n)$. By the hypothesis that A is invertible, a_0 is non-zero. On dividing by a_0 and changing some signs, we see that A satisfies a polynomial of the form $1 - c_0 t - c_1 t^2 - \cdots - c_{n-1}t^n$. This means that $I_n = c_0 A + c_1 A^2 + \cdots + c_{n-1}A^{n-1}$. Multiply by A^{-1} to get the desired expression.

Exhibit a 5×5 complex matrix whose characteristic polynomial is $t + 2t^2 + 3t^3 + 4t^4 - t^5$.

The answer is given by the displayed matrix on page 316 of the book. The matrix $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}$ seems to have the right characteristic polynomial. According to the book, the proof is represented by Exercise 19. As you remember, I worked out the proof in class during the last lecture before spring break.

4. *Let V be the real vector space $\mathcal{P}_3(\mathbf{R})$ of polynomials of degree ≤ 3 with real coefficients. Let $\beta = \{1, x, x^2, x^3\}$ be the standard ordered basis of V. Let $T : V \to V^*$ be the linear transformation that sends $p \in V$ to f_p, where $f_p(q) = \int_0^1 p(x)q(x)\,dx$. Find $[T]_\beta^{\beta^*}$, where β^* is the dual basis of β.*

To find $[T]_\beta^{\beta^*}$, we have to write $T(x^i)$ for $i = 0, \ldots, 3$ in terms of the basis β^*. It is natural to write $\beta = \{v_0, v_1, v_2, v_3\}$ and $\beta^* = \{f_0, f_1, f_2, f_3\}$, where $v_i = x^i$ ($i = 0, \ldots, 3$) and $f_j(v_i) = \delta_{ij}$. If f is an element of V^*, we have $f = \sum c_j f_j$ with $c_j = f(v_j)$ for each j. For each i, $T(v_i)$ is the linear form $q \mapsto \displaystyle\int_0^1 x^i q(x)\, dx$; the value of this form on v_j is $\displaystyle\int_0^1 x^i x^j\, dx = \frac{1}{i+j+1}$. Hence the matrix $[T]_\beta^{\beta^*}$ is the matrix (a_{ij}), $0 \le i, j \le 3$ with $a_{ij} = \dfrac{1}{i+j+1}$. This matrix is explicitly $\begin{pmatrix} 1/1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{pmatrix}$, I hope.