P. Vojta

Math 54M Final Exam

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1. (10 points) Write the matrix $\begin{bmatrix} 0 & 5 & 0 \\ 1 & 0 & 2 \\ -1 & 3 & -1 \end{bmatrix}$ as a product of elementary matrices.

2. (15 points) Define subspaces V and W in $\mathbb{R}^4$ by

$$V = \operatorname{Span}\{(1,2,3,4),(0,4,1,-1)\}, \qquad W = \operatorname{Span}\{(4,3,1,-6),(2,1,2,1)\}.$$

Find a nonzero vector in $V \cap W$.

3. (10 points) Let $\{\mathbf{u}, \mathbf{v}\}$ be a basis for a vector space W. Under what conditions on the scalars a, b, c, d is the set $\{a\mathbf{u} + b\mathbf{v}, c\mathbf{u} + d\mathbf{v}\}$ also a basis for W? Explain why.

4. (20 points) Find the least-squares solution to the following system of linear equations:

$$\begin{aligned} x + 3y + 5z &= -3 \\ 2x \qquad - 2z &= 5 \\ y + 2z &= 0 \\ x - y - 3z &= 7\,. \end{aligned}$$

If anything unusual occurs, explain it.

5. (20 points) Find the determinant.
Caution: Instruction #6 on the front page of the exam will be taken more seriously this time.

$$\begin{vmatrix} 2 & 0 & -3 & 4 \\ 1 & -1 & 0 & 1 \\ -3 & 5 & 4 & 7 \\ -5 & 1 & 11 & -3 \end{vmatrix} =$$

6. (15 points) The matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ has eigenvalues -1, 2, 2. Find an orthogonal matrix Q and a diagonal matrix Λ such that $Q^{-1}AQ = \Lambda$.

7. (10 points) Find the angle between the vector $(1, 2, 3)$ and the x-axis.

8. (15 points) Determine the longest interval in which the initial-value problem

$$(\tan t)y'' + (t-1)y' + 3y = \tan^2 t, \qquad y(1/2) = 0, \qquad y'(1/2) = 1$$

is certain to have a unique twice-differentiable solution. Do not attempt to find the solution.

9. (20 points) For the equation

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \mathbf{x} \,,$$

find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = I$.

10. (15 points) For the initial-value problem

$$\mathbf{x}' = \begin{bmatrix} 0 & -1 \\ 5 & -2 \end{bmatrix} \mathbf{x} \,, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \,,$$

describe the behavior of the solution as $t \to \infty$. (You do not need to solve the system.)

11. (20 points) The differential equation

$$\mathbf{x}' = \begin{bmatrix} 3 & -4 & 1 \\ -1 & 0 & -1 \\ -2 & 2 & 0 \end{bmatrix} \mathbf{x}$$

has a solution $\mathbf{x} = \begin{bmatrix} 6e^{2t} + e^{-t} \\ e^{-t} \\ -6e^{2t} \end{bmatrix}$. Find its general solution.

12. (25 points) Find the Fourier cosine series for the function $f(x) = \begin{cases} 1, & 0 \le x \le 1/2 \\ 0, & 1/2 < x \le 1 \end{cases}$.

13. (30 points) (a). Use the method of separation of variables to replace the partial differential equation

$$u_{tt} - 2u_t + u_{xx} = 0, \qquad u(0,t) = u(1,t) = 0$$

with a pair of ordinary differential equations.

(b). Find a set of fundamental solutions for the above PDE. You are allowed (and in fact encouraged) to use remembered facts about the heat equation.