MATH 126 — MIDTERM

Problem #1. Suppose $u = u(x,t)$ solves the first-order linear PDE:

$$\begin{cases} u_t + 6u_x = 0 & \text{for } -\infty < x < \infty, t \geq 0 \\ u(x,0) = \cos x & \text{for } -\infty < x < \infty. \end{cases}$$

Find an explicit formula for u.

Problem #2. Let $u = X(x)T(t)$ solve the PDE

$$u_{xt} = u_{xx} + u.$$

What ODEs do the functions X and T satisfy?

Problem #3. Suppose that $u = u(x,t)$ solves the heat equation

$$\begin{cases} u_t = u_{xx} & \text{for } 0 < x < l,\ t \geq 0 \\ u = 0 & \text{for } x = 0 \text{ and } l,\ t \geq 0. \end{cases}$$

Show that

$$\frac{d}{dt}\int_0^l u^2(x,t)\,dx \leq 0.$$

Problem #4. Assume $u = u(x,t)$ solves the wave equation

$$\begin{cases} u_{tt} = u_{xx} & \text{for } -\infty < x < \infty,\ t \geq 0 \\ u = \phi, u_t = \psi & \text{for } -\infty < x < \infty,\ t = 0. \end{cases}$$

Use the following procedure to prove that if both ϕ and ψ are odd, then u is odd in the variable x for times $t \geq 0$.

Let

$$\hat{u}(x,t) = -u(-x,t),$$

and show that $\hat{u}$ solves the same PDE as u, with the same initial conditions. Why does this imply $\hat{u} \equiv u$?

Problem #5. Assume $u = u(x,t)$ solves the wave equation

$$\begin{cases} u_{tt} = u_{xx} & \text{for } -\infty < x < \infty,\ t \geq 0 \\ u = \phi, u_t = \psi & \text{for } -\infty < x < \infty,\ t = 0. \end{cases}$$

Derive d'Alembert's formula for the solution, using the following hints.

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First, you may assume that u has the form

$$u(x,t) = f(x+t) + g(x-t).$$

Evaluate $u(x,0)$ and $u_t(x,0)$ in terms of f and g, to get the ODE

$$f' + g' = \phi', \quad f' - g' = \psi.$$

Now solve for f' and g' and then integrate, to compute f, g in terms of ϕ, ψ.

Problem #6. Suppose $v = v(x,t)$ solves the nonlinear PDE

$$\begin{cases} v_t = v_{xx} + v_x^2 & \text{for } -\infty < x < \infty,\ t \geq 0 \\ v = \phi & \text{for } -\infty < x < \infty,\ t = 0. \end{cases}$$

Use the following hints to derive a formula for v.

First, show that $u = e^v$ solves the heat equation

$$\begin{cases} u_t = u_{xx} & \text{for } -\infty < x < \infty,\ t \geq 0 \\ u = e^\phi & \text{for } -\infty < x < \infty,\ t = 0. \end{cases}$$

Use the fundamental solution to write down a formula for u, and from this determine a formula for v.