Mathematics 54
Midterm #2 Spring 2009
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EXAM QUESTIONS

Problem # 1 Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}$

(A) Rank(A) = 1 , Nullity(A) = 4
(B) Rank(A) = 4 , Nullity(A) = 1
(C) Rank(A) = 3 , Nullity(A) = 2

Problem #2 Let u_1 and u_2 be eigenvectors of a 5×5 matrix A, and let λ_1 and λ_2 be the corresponding eigenvalues. If u_1 and u_2 are linearly independent, then $\lambda_1 \neq \lambda_2$.

(A) True **(B)** False

Problem #3 The matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$
is diagonalizable .

(A) True **(B)** False

Problem #4 Let R^4 be given the standard inner product coming from dot product of vectors. Let $\{x_1, x_2, x_3\}$ be three linearly independent vectors in R^4, and let $\{v_1, v_2, v_3\}$ be the set of orthogonal vectors coming from $\{x_1, x_2, x_3\}$ by the Gram - Schmidt process. Then

$$v_1 = x_1$$

$$v_2 = x_2 - \left(\frac{x_1 \bullet v_1}{v_1 \bullet v_1}\right) v_1$$

$$v_3 = x_3 - \left(\frac{x_2 \bullet v_1}{v_1 \bullet v_1}\right) v_1 - \left(\frac{x_2 \bullet v_2}{v_1 \bullet v_2}\right) v_2$$

(A) True **(B)** False

Problem #5 Let A be a 3×3 symmetric matrix with characteristic polynomial $\chi_A(\lambda) = -(\lambda - 1)(\lambda - 5)^2$. Which of the following is true?

(A) Every basis of eigenvectors for A is an orthogonal set of vectors.

(B) There is a basis of eigenvectors for A which is not an orthogonal set of vectors.

Problem #6 Let A be a 5×5 matrix with a basis of eigenvectors $\{v_1, v_2, v_3, v_4, v_5\}$. Consider the matrix $Q = [v_1\ v_2\ v_3\ v_4\ v_5]$. Which of the following matrices is always diagonal ?

(A) QAQ^{-1} **(B)** Q^TAQ **(C)** $Q^{-1}AQ$ **(D)** None of these.

Problem #7 Let ϵ denote the standard basis for R^2 and let $\alpha = \{u_1, u_2\}$ be the basis for R^2 where

$$u_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \quad \text{and} \quad u_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The change of basis matrix P_ϵ^α is

(A) $\begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$ **(B)** $\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$ **(C)** Neither (A) nor (B).

Problem #8. Consider the basis $\alpha = \{v_1, v_2, v_3\}$ for R^3 where

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$. Let $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The coordinate vector $[x]_\alpha$ of x with respect to α is

(A) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ **(B)** $\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ **(C)** $\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$ **(D)** None of these.

Problem #9 Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 5 \end{bmatrix}$ and let $b = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$.

(9.1) One least squares solution $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to the linear equations $Ax = b$ is obtained by solving the linear equations $Ax = c$ where c is

(A) $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ **(B)** $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ **(C)** $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

(9.2) The normal equations for this problem are

$$\begin{aligned} 2x_1 + 12x_2 &= 4 \\ 12x_1 + 27x_2 &= 11 \end{aligned}$$

(A) True **(B)** False

(9.3) There is only one least squares solution x to this problem.

(A) True **(B)** False

Problem #10 Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and define an inner product on R^2 by the formula $< x, y >= Ax \bullet Ay$ where the right hand side is the usual dot product in R^2. Let $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $W = Span\{u\}$. Let $W^\perp$ denote the subspace of all vectors in R^2 which are orthogonal to W.

(10.1) $W^\perp$ consists of all vectors $v = \begin{bmatrix} x \\ y \end{bmatrix}$ where

(A) $x = y$ **(B)** $x = 0$ **(C)** $x = -y$

(10.2) The matrix representation of the orthogonal projection P_W onto W with respect to the standard basis is

(A) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

Problem #11 Consider the subspace $W = Span\{u_1, u_2\}$ of R^3 where

$u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. We also have $W = Span\{v_1, v_2\}$ where $v_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1/\sqrt{2} \end{bmatrix}$. Consider the bases $\alpha = \{u_1, u_2\}$ and $\beta = \{v_1, v_2\}$ for W. The change of basis matrix P_α^β is

(A) $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 \end{bmatrix}$ (C) $\begin{bmatrix} 1/2\sqrt{2} & 1/2\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/2 \\ -1/2 & -1/2 & 1/\sqrt{2} \end{bmatrix}$

Problem #12 Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ and the basis

$\alpha = \{v_1, v_2, v_3\}$ for R^3 where

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

(12.1) The matrix representation $[A]_\alpha^\alpha$ of A

with respect to α is

(A) $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ (C) $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(12.2) The characteristic polynomial $\chi_A(\lambda)$ of A is

(A) $(1-\lambda)(2-\lambda)(3-\lambda)$ **(B)** $-\lambda(6-\lambda)^2$ **(C)** $\lambda^2(6-\lambda)$

Problem #13 Let A be a 5×5 matrix which satisfies the matrix equation $A^3 = 4A^2 + 5A$. The eigenvalues for A are

(A) $\lambda = 0, -1, 5$ **(B)** $\lambda = 0, 1, -5$ **(C)** Neither (A) nor (B).