Midterm II, November 16,

Work each problem on a separate sheet of paper. Be sure to put your name, your section number, and your GSI's name on each sheet of paper. Also, at the top of the page, in the center, write the problem number, and be sure to put the pages in order. Write clearly: explanations (with complete sentences when appropriate) will help us understand what you are doing. Note that there are problems on the back of this sheet, for a total of four problems.

1. Let $A := \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

   (a) (10 pts) Find the eigenvalues of A.

   (b) (10 pts) For each eigenvalue, give a basis for the corresponding eigenspace.

   (c) (5 pts) Is A diagonalizable?

   Notes: Explain your work step by step. Partial credit will only be given if your writing is very clear and easy to follow, with little or no partial credit for answers which would have been correct based on a mistake in an early step. If you copy the matrix wrong at the beginning, you will get no credit.

   Solution:

   (a) We compute the characteristic polynomial of A:

   $$f_A(t) := \det(A - tI) = \begin{vmatrix} 1-t & -1 & 0 \\ 0 & 2-t & 1 \\ 0 & 1 & 2-t \end{vmatrix}$$

   $$= (1-t)\left|\begin{pmatrix} 2-t & 1 \\ 1 & 2-t \end{pmatrix}\right| = (1-t)(t^2-4t+3) = -(t-3)(t-1)^2.$$

   Thus the eigenvalues roots are 1 and 3, with multiplicities 2 and 1, respecitively.

   (b)
   $$Eig_3(A) = NS(A - 3I)$$

   $$= NS\begin{pmatrix} -2 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} = NS\begin{pmatrix} -2 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = span\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

   $$Eig_1(A) = NS(A-I) = NS\begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = NS\begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = span\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(c) This matrix is not diagonalizable because the algebraic multiplicty of the eigenvalue 1 is 2, but its geometric multiplicity is 1.

2. Answer the following questions using complete sentences.

   (a) (5 pts) What does it mean for two matrices to be *similar*?

   (b) (5 pts) Suppose that A and B are similar matrices. Is it true that every eigenvalue of A is an eigenvalue of B? Give a proof or a counterexample.

   (c) (5 pts) Suppose that A and B are similar matrices. Is it true that every eigenvector of A is an eigenvector of B? Give a proof or a counterexample.

   (d) (5 pts) Suppose that A and B are similar matrices and $A^2 + 2A = 3I$. Is it true that $B^2 + 2B = 3I$? Give a proof or a counterexample.

   Solutions:

   (a) Two matrices A and B are similiar if there exists an invertible matrix S such that $B = S^{-1}AS$.

   (b) Yes. If $v \neq 0$ and $Av = \lambda v$, then $w := S^{-1}v \neq 0$, and

   $$Bw = S^{-1}ASS^{-1}v = S^{-1}Av = S^{-1}\lambda v = \lambda w.$$

   (c) This is not true. For example, the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector of $A := \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, but not of the matrix $B := S^{-1}AS$, where for example $S = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

   (d) This is true:

   $$3I = S^{-1}(3I)S = S^{-1}(A^2+2A)S = S^{-1}(A^2)S+S^{-1}(2A)S = B^2+2B$$

3. Let V be the vector space consisting of polynomials of degree less than or equal to 2, and let $T\colon V \to V$ be the map sending f to $xf' + f''$.

   (a) (5 pts) Find the eigenvalues of T.

   (b) (10 pts) If possible, find a basis for V consisting of eigenvectors for T.

   Solutions:

   i. The matrix for T with respect to the basis $(1, x, x^2)$ is $\begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

      Thus the eigenvalues are $\{0, 1, 2\}$.

   ii. Solution: $(1, x, x^2 + 1)$

4. (a) (5 pts) Use the Gram-Schmidt procedure to find an orthogonal basis for the column space of the matrix $\begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$

Solution: Let v_1 and v_2 be the columns. Then $w_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$,

and

$$w_2 = v_2 - \frac{(w_1 \cdot v_2)}{(w_1 \cdot w_1)} w_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \frac{3}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

(b) You are told to solve the system of equations

$$\begin{aligned} x_1 + 2x_2 &= 2 \\ x_1 &= 4 \\ x_1 + x_2 &= 0 \end{aligned}$$

After hearing your explanation that this is impossible, your boss (that's me) tells you to find x_1 and x_2 so that the vector

$$\mathbf{y}^* := \begin{pmatrix} x_1 + 2x_2 \\ x_1 \\ x_1 + x_2 \end{pmatrix}$$

is as close as possible to the vector $\mathbf{y} := \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$.

i. (10 pts) What is your answer for $\mathbf{x}^* := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$?

ii. (5 pts) What is the your answer for $\mathbf{y}^*$?

iii. (5 pts) What is the distance between $\mathbf{y}^*$ and $\mathbf{y}$?

Solution method 1. Use the normal equation:

$$A^T A \mathbf{x}^* = A^T \mathbf{y}.$$

Then

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix},$$

so

$$(A^T A)^{-1} = \frac{1}{6} \begin{pmatrix} 5 & -3 \\ -3 & 3 \end{pmatrix}.$$

Furthermore,

$$A^T \mathbf{y} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}.$$

Thus

i. $\mathbf{x}^* = (A^T A)^{-1} A^T \mathbf{y} = \frac{1}{6} \begin{pmatrix} 5 & -3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

ii. $\mathbf{y}^* = A\mathbf{x}^* \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

iii. $||\mathbf{y} - \mathbf{y}^*|| = || \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} || = \sqrt{6}$

Solution method 2: We use the fact that $\mathbf{y}^*$ is the orthogonal projection onto the column space, which can be computed using the orthogonal basis found in the previous part:

$$\mathbf{y}^* = \frac{\mathbf{y} \cdot w_1}{w_1 \cdot w_1} w_1 + \frac{\mathbf{y} \cdot w_2}{w_2 \cdot w_2} w_2 = \frac{6}{3} w_1 + \frac{-2}{2} w_2 = 2w_1 - w_2$$

$$= 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

This is the answer to part ii. To get the answer to part i, we must write $\mathbf{y}^*$ as a linear combination of v_1 and v_2. But $w_1 = v_1$ and $w_2 = v_2 - v_1$, so

$$\mathbf{y}^* = 2w_1 - w_2 = 2v_1 - (v_2 - v_1) = 3v_1 - v_2$$

Hence $x_1 = 3$ and $x_2 = -1$.