Math 54 - Midterm #2

August 2, 2004, 10:00am-12:00pm

Name: Solutions

This is a closed book, closed notes exam. Calculators are not allowed. You have two hours to complete the exam. To receive full credit, write legibly, show your work and write proofs in complete sentences. If you need more space, use the back of the page of the problem on which you are working.

Problem	Points	Your Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
Total	120	

1. Let

$$A = \begin{bmatrix} 3 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

(a) Find the eigenvalues of A.

(b) Find bases for the eigenspaces of A.

(c) Is A diagonalizable? If so, diagonalize A.

(a) We want to find λ such that $0 = \det(A - \lambda I)$

$$= \det \begin{bmatrix} 3-\lambda & -1 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 1 & 3-\lambda \end{bmatrix} = (2-\lambda)\det\begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} = (2-\lambda)[(3-\lambda)^2 - 1]$$

$$= (2-\lambda)(\lambda^2 - 6\lambda + 8) = (2-\lambda)(\lambda-2)(\lambda-4) = -(\lambda-2)^2(\lambda-4),$$

so the eigenvalues of A are $2, 2, 4$.

(b) We calculate $W_2(A) = NS(A - 2I) = NS\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

$$= NS\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\},$$ and $W_4(A) = NS(A - 4I)$

$$= NS\begin{bmatrix} -1 & -1 & -1 \\ 0 & -2 & 0 \\ -1 & 1 & -1 \end{bmatrix} = NS\begin{bmatrix} -1 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & 2 & 0 \end{bmatrix} = NS\begin{bmatrix} -1 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= \text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, so $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ are bases for $W_2(A)$ and $W_4(A)$, respectively.

(c) A is diagonalizable since it has three linearly independent eigenvectors, and if $S = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ then $S^{-1}AS = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

2. Let $W = \text{span}\{(1,-1,0,1),(2,1,0,0)\} \subseteq \mathbb{R}^4$.

(a) Find an orthonormal basis for W.

(b) Find the point $x \in W$ that is closest to $y = (2,-1,1,3)$.

(a) Let $v_1 = (1,-1,0,1)$, $v_2 = (2,1,0,0)$. We perform Gram-Schmidt on $\{v_1, v_2\}$: Let $u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}(1,-1,0,1)$. Let $w_2 = v_2 - \langle v_2, u_1 \rangle u_1$

$$= (2,1,0,0) - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}(1,-1,0,1) = (2,1,0,0) - \frac{1}{3}(1,-1,0,1)$$

$= \frac{1}{3}(5,4,0,-1)$. Let $u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{\sqrt{42}}(5,4,0,-1)$.

So $\{u_1, u_2\} = \{\frac{1}{\sqrt{3}}(1,-1,0,1), \frac{1}{\sqrt{42}}(5,4,0,-1)\}$ is an orthonormal basis for W.

(b) We'd like to find $\alpha, \beta \in \mathbb{R}$ so that $\alpha(1,-1,0,1) + \beta(2,1,0,0) = (2,-1,1,3)$, or $A\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = b$, where $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{bmatrix}$. This isn't possible, so we'll ~~solve~~ find ~~the~~ $\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ so that $A\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$ is close to b.

To do this, we solve the normal equations:

$$A^T A \begin{bmatrix}\alpha\\ \beta\end{bmatrix} = A^T b, \text{ or } \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}.$$

We use Gaussian elimination:

$$\left[\begin{array}{cc|c} 3 & 1 & 6 \\ 1 & 5 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 5 & 3 \\ 3 & 1 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 5 & 3 \\ 0 & -14 & -3 \end{array}\right], \text{ so } \beta = \frac{3}{14},$$

$\alpha = 3 - 5\beta = 3 - \frac{15}{14} = \frac{27}{14}$. Therefore the closest point $x \in W$ to y is $x = \frac{27}{14}(1,-1,0,1) + \frac{3}{14}(2,1,0,0) = \left(\frac{33}{14}, -\frac{12}{7}, 0, \frac{27}{14}\right)$.

3. Find the general solution of the ODE

$$f'' + 2f' + 6f = 0 \tag{1}$$

and find solutions f_1 and f_2 of (1) such that $f_1(0) = 1$, $f_1'(0) = -1$, $f_2(0) = 2$, and $f_2'(0) = 1$. Is $\{f_1, f_2\}$ a fundamental set of solutions for (1)?

First we solve the characteristic equation $r^2 + 2r + 6 = 0$:

$$r = \frac{-2 \pm \sqrt{4-24}}{2} = -1 \pm \sqrt{5}\, i$$

Thus the general solution is

$$f(t) = c_1 e^{-t} \cos \sqrt{5} t + c_2 e^{-t} \sin \sqrt{5} t .$$

To find f_1: $\quad 1 = f_1(0) = c_1,$

$-1 = f_1'(0) = -c_1 + \sqrt{5} c_2$, so $c_1 = 1$, $c_2 = 0$.

So $f_1(t) = e^{-t} \cos \sqrt{5} t$.

To find f_2: $\quad 2 = f_2(0) = c_2,$

$1 = f_2'(0) = -c_1 + \sqrt{5} c_2$, so $c_1 = 2$, $c_2 = \frac{3\sqrt{5}}{5}$.

So $f_2(t) = 2e^{-t} \cos \sqrt{5} t + \frac{3\sqrt{5}}{5} e^{-t} \sin \sqrt{5} t$.

Since $W(f_1, f_2)(0) = f_1(0) f_2'(0) - f_2(0) f_1'(0)$

$= (1)(1) - (2)(-1) = 3 \neq 0,$

$\{f_1, f_2\}$ is a fundamental set for (1).

4. For each of the following, determine if A is definitely diagonalizable, maybe or maybe not diagonalizable, or definitely not diagonalizable. Explain your answers.

(a) $A = \begin{bmatrix} -1 & -1 & 2 \\ 0 & -3 & -1 \\ 0 & 0 & 2 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 4 & -1 & 2 \\ -1 & 1 & 6 \\ 2 & 6 & -2 \end{bmatrix}$.

(c) $A = B^T B$ for some $m \times n$ matrix B.

(d) $A = BC$, where B and C are square, diagonalizable matrices.

(a) Since A is upper triangular, its eigenvalues are on the diagonal: $-1, -3, 2$. Since A has three distinct eigenvalues, it is diagonalizable.

(b) Since A is symmetric, A is diagonalizable.

(c) Since $A^T = (B^T B)^T = B^T (B^T)^T = B^T B = A$, A is diagonalizable.

(d) A may or may not be diagonalizable. For instance, if $B = C = I$ then $A = I$ is diagonalizable. If

$$B = \begin{bmatrix} 2 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} \text{ and } C = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 2 \end{bmatrix}, \text{ then } A = BC = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is not diagonalizable although B and C are.

5. For each of the following, find (or show there does not exist) a square matrix B with real entries such that $B^4 = A$.

(a) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$

(a) The eigenvalues of A are $3, 1$, and $W_1(A) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$, $W_3(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$. Let $S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Then $S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, and $A = S\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}S^{-1}$. Let $B = S\begin{bmatrix} 1 & 0 \\ 0 & \sqrt[4]{3} \end{bmatrix}S^{-1}$. Then $B^4 = S\begin{bmatrix} 1 & 0 \\ 0 & \sqrt[4]{3} \end{bmatrix}^4 S^{-1} = A$.

So we compute $B = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt[4]{3} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ \sqrt[4]{3} & \sqrt[4]{3} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\sqrt[4]{3} & -1+\sqrt[4]{3} \\ -1+\sqrt[4]{3} & 1+\sqrt[4]{3} \end{bmatrix}.$$

(b) If $B^4 = A$, then $(\det B)^4 = \det B^4 = \det A = -1$, which is impossible. Therefore no such B exists.

6. Let A, B and S be $n \times n$ matrices with S invertible.

(a) Prove that A and $S^{-1}AS$ have the same eigenvalues.

(b) Prove that if A is invertible then AB and BA have the same eigenvalues.

(a) Since $\det S^{-1} = \dfrac{1}{\det S}$,

$$\det(A - \lambda I) = \det S^{-1} \det(A - \lambda I) \det S$$
$$= \det[S^{-1}(A - \lambda I)S] = \det(S^{-1}AS - \lambda I).$$

Thus A and $S^{-1}AS$ have the same characteristic polynomial and hence the same eigenvalues.

(b) Since $BA = A^{-1}(AB)A$, BA and AB have the same eigenvalues by part (a).