EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 1
Department of Electrical Engineering and Computer Sciences 18 September 2007
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name Quincy FIRST Name Free
Lab Time Lab? What lab?

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided 8.5" × 11" sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 10.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the ten numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

Formulas and Facts of Potential Use or Interest:

Trigonometric Identities:

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta \\ \cos\alpha\,\cos\beta &= \frac{1}{2}\Big[\cos(\alpha+\beta)+\cos(\alpha-\beta)\Big]\end{aligned}$$

Integral of an Exponential: For $\alpha, \beta \in \mathbb{R} \cup \{-\infty, +\infty\}$ and $\lambda \in \mathbb{C}$,

$$\int_\alpha^\beta e^{\lambda t}dt = \frac{e^{\beta\lambda} - e^{\alpha\lambda}}{\lambda}.$$

Even-and-Odd Decomposition of a Function A function $x : \mathbb{R} \to \mathbb{R}$ can be decomposed into the sum of an even function and an odd function as follows:

$$x(t) = x_e(t) + x_o(t) \quad \forall t,$$

where x_e and x_o denote the even and odd components, respectively, and are related to x according to the equations below:

$$x_e(t) = \frac{x(t)+x(-t)}{2} \qquad x_o(t) = \frac{x(t)-x(-t)}{2}.$$

Sum of Sinusoids of the Same Frequency: Given $A_k > 0$ and $\phi_k \in \mathbb{R}$, there exist $A > 0$ and $\phi \in \mathbb{R}$ such that

$$A\cos(\omega t+\phi) = \sum_{k=1}^{N} A_k \cos(\omega t + \phi_k),$$

where

$$\begin{aligned} A &= \left[\left(\sum_{k=1}^{N} A_k\cos\phi_k\right)^2 + \left(\sum_{k=1}^{N} A_k\sin\phi_k\right)^2\right]^{1/2} \\ \phi &= \tan^{-1}\left(\frac{\sum_{k=1}^{N} A_k\sin\phi_k}{\sum_{k=1}^{N} A_k\cos\phi_k}\right).\end{aligned}$$

Combinations and Permutations

Combinations: $\begin{pmatrix} N \\ M \end{pmatrix} = \dfrac{N!}{M!\,(N-M)!}$ Permutations: $\dfrac{N!}{(N-M)!}.$

MT1.1 (30 Points) The continuous-time signal x characterized by

$$\forall t \in \mathbb{R}, \quad x(t) = A\cos(\omega t + \phi)$$

is ubiquitous in the engineering and physical sciences. The parameters $A > 0$ and $\phi \in \mathbb{R}$ denote the amplitude and phase of x, respectively.

The two parts of this problem are mutually independent and can be tackled in either order.

(a) Determine simple expressions for $x_e(t)$ and $x_o(t)$ in terms of A, ϕ, ω, and t, where x_e and x_o denote the even and odd components of x, respectively.

By way of a "sanity check," interpret (and justify) your answers for $\phi = 0$ and $\phi = -\pi/2$. In particular, explain how x_e and x_o relate to x for these two values of ϕ.

$$x(t) = A\cos(\omega t + \phi) = \underbrace{A\cos\phi\cos\omega t}_{x_e(t)} + \underbrace{(-A\sin\phi\sin\omega t)}_{x_o(t)}$$

By inspection of the right-hand side we recognize that

$$x_e(t) = A\cos\phi\cos\omega t \quad \text{and} \quad x_o(t) = -A\sin\phi\sin\omega t$$

based on the understanding that $\cos\omega t$ is an even function of t, and $\sin\omega t$ is an odd function of t.

Sanity Check: $\phi = 0 \Rightarrow \begin{cases} x(t) = A\cos\omega t \\ x_e(t) = A\cos 0\cos\omega t = A\cos\omega t \\ x_o(t) = -A\sin 0\sin\omega t = 0 \end{cases}$ ✓

$\phi = -\frac{\pi}{2} \Rightarrow \begin{cases} x(t) = A\cos(\omega t - \frac{\pi}{2}) = A\sin\omega t \\ x_e(t) = A\cos(-\frac{\pi}{2})\cos\omega t = 0 \\ x_o(t) = -A\sin(-\frac{\pi}{2})\sin\omega t = A\sin\omega t \end{cases}$ ✓

(b) We can express a sum of sinusoids of frequency ω as a single sinusoid of frequency ω. Determine numerical values for A and ϕ in the equation below, where the right-hand side is a sum of two sinusoids of frequency ω:

$$A\cos(\omega t + \phi) = \cos \omega t + \sqrt{3} \sin \omega t. \qquad (\star)$$

Your trigonometry teacher rightfully used to insist that

$$\cos\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad \text{and} \quad \sin\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.$$

Recast the right-hand side of Equation $(\star)$ as a sum of cosines:

$$A\cos(\omega t + \phi) = \cos \omega t + \sqrt{3}\cos\left(\omega t - \frac{\pi}{2}\right)$$

This looks like a special case of the sum of two cosine waves of identical frequencies but different phases:

$$\begin{cases} A_1 = 1 \\ \phi_1 = 0 \\ A_2 = \sqrt{3} \\ \phi_2 = -\pi/2 \end{cases}$$

$$A\cos(\omega t + \phi) = A_1 \cos(\omega t + \phi_1) + A_2 \cos(\omega t + \phi_2)$$

Let $Ae^{i\phi}$ be the polar representation of the sum $A_1 e^{i\phi_1} + A_2 e^{i\phi_2}$. That is, let $Ae^{i\phi} = A_1 e^{i\phi_1} + A_2 e^{i\phi_2}$.

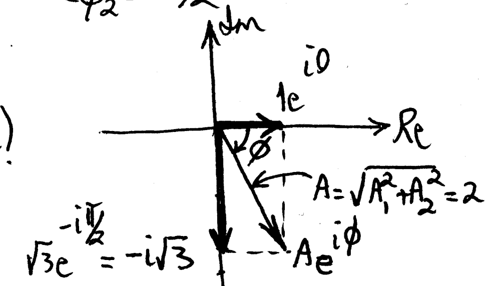
Multiply each side by $e^{i\omega t}$: $Ae^{i(\omega t + \phi)} = A_1 e^{i(\omega t + \phi_1)} + A_2 e^{i(\omega t + \phi_2)}$

Equate the real parts of the two sides:

$$A\cos(\omega t + \phi) = A_1\cos(\omega t + \phi_1) + A_2\cos(\omega t + \phi_2)$$

For our particular case, $Ae^{i\phi} = 1e^{i0} + \sqrt{3}e^{-i\pi/2}$

We find A and ϕ vectorially, as in the diagram to the right.



$$\phi = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\pi/3$$

You may use the blank space below for scratch work. Nothing written below this line on this page will be considered in evaluating your work.

Sanity Check: $A = 2$, $\phi = \frac{-\pi}{3} \Rightarrow A\cos(\omega t + \phi) = 2\cos\left(\omega t - \frac{\pi}{3}\right)$

$$\Rightarrow A\cos(\omega t + \phi) = 2\left[\cos \omega t \underbrace{\cos\left(\frac{-\pi}{3}\right)}_{1/2} - \sin \omega t \underbrace{\sin\left(\frac{-\pi}{3}\right)}_{-\frac{\sqrt{3}}{2}}\right] = 2\left(\frac{\cos \omega t}{2} + \frac{\sqrt{3}\sin \omega t}{2}\right)$$

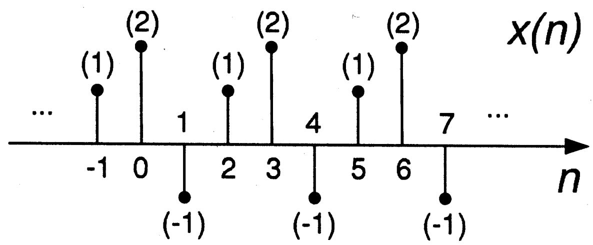
$$\Rightarrow A\cos(\omega t + \phi) = 2\cos\left(\omega t - \frac{\pi}{3}\right) = \cos \omega t + \sqrt{3}\sin \omega t \quad \checkmark$$

MT1.2 (25 Points) A discrete-time signal x is said to be *periodic* if there exists an integer $p > 0$ such that

$$\forall n \in \mathbb{Z}, \quad x(n+p) = x(n). \tag{1}$$

The smallest p for which Equation (1) holds is called the *fundamental period* of x.

An example of a periodic discrete-time signal (with period $p = 3$) is shown below:



The signal in the figure is illustrative only. In this problem, you should *not* assume that the signal x is as shown in the figure, *nor* that it has fundamental period $p = 3$.

It is known that the signal x can be decomposed into a linear combination of discrete-time complex exponential functions according to the following equation:

$$\forall n \in \mathbb{Z}, \quad x(n) = \sum_{k=0}^{M} X_k \, e^{ik\frac{2\pi}{p}n}, \tag{2}$$

where X_k is the k^{th} coefficient of the linear combination. Do not worry about how this expansion is obtained; you will learn all about that later in the course. For now, assume Equation (2) is true for some appropriate value of M. Here, we explore some of the properties of the expansion depicted by Equation (2).

(a) Show that the signal x expressed in the form of Equation (2) is periodic with period p. That is, show $x(n+p) = x(n), \forall n \in \mathbb{Z}$.

$$x(n+p) = \sum_{k=0}^{M} X_k e^{ik\frac{2\pi}{p}(n+p)} = \sum_{k=0}^{M} X_k e^{ik\frac{2\pi}{p}n} e^{ik\frac{2\pi}{\cancel{p}}\cancel{p}}$$

$$= \sum_{k=0}^{M} X_k e^{ik\frac{2\pi}{p}n} \underbrace{\cancel{e^{i2\pi k}}}_{1} = \sum_{k=0}^{M} X_k e^{ik\frac{2\pi}{p}n} = x(n) \quad , \forall n \in \mathbb{Z}$$

We've shown that $x(n+p) = x(n)$ $\forall n \in \mathbb{Z}$ using the fact that $e^{i2\pi k} = 1$, $\forall k \in \mathbb{Z}$.

(b) The ratio $2\pi/p$ in Equation (2) is called the *fundamental frequency* of x and is denoted by ω_0. Each discrete-time complex exponential function $e^{ik\frac{2\pi}{p}n}$ represents signal content at frequency $k\omega_0$.

Equation (2) states that the signal x contains *only* integer multiples of the fundamental frequency ω_0. In particular, x contains the following frequencies:

$$0, \omega_0, 2\omega_0, \ldots, M\omega_0.$$

Determine the largest number of *distinct* frequencies that can be present in x, and from that infer the largest possible value for M. Could M be infinitely large?

Method 1:

There can be at most p distinct frequencies in the signal x. To see this, recall that $e^{i\frac{2\pi}{p}}$ is a p^{th} root of unity. Well, so is $e^{i\frac{2\pi}{p}n}$ (raise $e^{i\frac{2\pi}{p}n}$ to the p^{th} power if you're skeptical). We know that there are exactly p distinct powers of any p^{th} root of unity: $\left(e^{i\frac{2\pi}{p}n}\right)^k$ $k=0,1,\cdots,p-1 \Rightarrow \boxed{M=p-1}$

Method 2:

You can also arrive at the same result using what you discovered in part (a) about the periodicity (in terms of k or n) of the discrete-time complex exponential: $e^{ik\frac{2\pi}{p}n} = e^{i(k+p)\frac{2\pi}{p}n}$.

You may use the blank space below for scratch work. Nothing written below this line on this page will be considered in evaluating your work.

Side Note: We say x contains "at most" p distinct frequencies (in contrast to saying x contains "exactly" p frequencies) because if the coefficient X_k of the k^{th} harmonic frequency term $e^{ik\frac{2\pi}{p}n}$ is zero, then that frequency $k\frac{2\pi}{p} \triangleq k\omega_0$ will not be present in the signal x.

The k^{th} harmonic frequency is the k^{th} multiple of the fundamental frequency ω_0: $k\omega_0$ is the k^{th} harmonic.

MT1.3 (25 Points) A reasonable measure of the "strength" or "size" of a continuous-time signal $x : \mathbb{R} \to \mathbb{C}$ accounts for not only the signal's amplitude variations but also its duration. An example of such a measure is the *total energy* of x, which is defined by

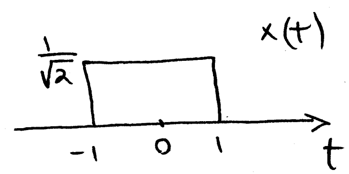
$$\mathcal{E}_x = \int_{-\infty}^{+\infty} x(t)\, x^*(t)\, dt \triangleq \int_{-\infty}^{+\infty} |x(t)|^2\, dt\,.$$

(a) Determine the energy of the signal x defined by

Draw pictures whenever you can.

$$\forall t \in \mathbb{R}, \quad x(t) = \begin{cases} 1/\sqrt{2} & |t| \le 1 \\ 0 & |t| > 1. \end{cases}$$

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-1}^{1} \left(\frac{1}{\sqrt{2}}\right)^2 dt = \frac{1}{2}\int_{-1}^{1} dt = \frac{1}{2}(2) = 1$$



$$\mathcal{E}_x = 1$$

(b) Determine the energy of the signal $x : \mathbb{R} \to \mathbb{C}$, where $x(t) = e^{s|t|}, \forall t \in \mathbb{R}$. The variable s is complex, so it can be written in Cartesian form as $s = \sigma + i\omega$. Assume $\sigma < 0 < \omega$.

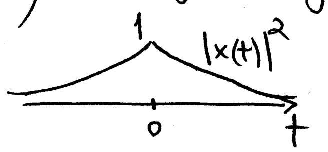
(i) Determine $\mathcal{E}_x$, the total energy of the signal x.

$x(t) = e^{(\sigma + i\omega)|t|}$

Method 1: $\mathcal{E}_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \left|e^{(\sigma+i\omega)|t|}\right|^2 dt = \int_{-\infty}^{\infty} \left|e^{\sigma|t|}\right|^2 \underbrace{\cancel{\left|e^{i\omega|t|}\right|^2}}_{1} dt = \int_{-\infty}^{\infty} e^{2\sigma|t|} dt = 2\int_{0}^{\infty} e^{2\sigma t} dt$ By symmetry

$\Rightarrow \mathcal{E}_x = \cancel{2}\, \frac{e^{2\sigma t}}{\cancel{2}\sigma}\Big|_0^{\infty} = -\frac{1}{\sigma} \Rightarrow \mathcal{E}_x = -\frac{1}{\sigma}$ (Note: $\sigma < 0 \Rightarrow \mathcal{E}_x > 0$ as expected)

Method 2: $x^*(t) = e^{(\sigma - i\omega)|t|} \Rightarrow x(t)x^*(t) = e^{2\sigma|t|} \Rightarrow |x(t)|^2 = x(t)x^*(t) = e^{2\sigma|t|}$



The rest is the same as the last portion of Method 1.

(ii) Explain how σ and ω affect the total energy of x. In particular, explain what happens to $\mathcal{E}_x$ as $\sigma \to -\infty$ and as $\sigma \to 0$.

σ is a measure of the decay of $x(t)$ as $|t| \to \infty$. The faster the signal decays, the lower its total energy. The signal decays faster as σ becomes more and more negative (moving toward $-\infty$). It decays more slowly as σ becomes less and less negative (moving toward 0 from the left). The limiting cases are: $\lim_{\sigma \to -\infty} \mathcal{E}_x = 0$ and $\lim_{\sigma \to 0^-} \mathcal{E}_x = +\infty$.

Note that when $\sigma \to 0^-$, the signal approaches a non-decaying complex exponential, which has infinite energy (finite energy in each period, added over an infinite number of periods). The frequency ω does not affect the total energy, because $\left|e^{i\omega|t|}\right| = 1 \;\; \forall t$.

MT1.4 (25 Points) Consider a digital circuit that takes a five-digit user input and produces a five-digit codeword.

In particular, the user's input is a string of five binary digits $x_1x_2x_3x_4x_5$, where $x_k \in \{0,1\}, k = 1, \ldots 5$.

The codeword, on the other hand, is a string of five hexadecimal digits $y_1y_2y_3y_4y_5$, where $y_\ell \in \{0,1,\ldots,9,A,B,C,D,E,F\}, \ell = 1,\ldots 5$.

The circuit can be partially described by the function

$$f : \{0,1\}^5 \to \{0,1,\ldots,9,A,B,C,D,E,F\}^5.$$

(a) Making no assumption about whether an arbitrary user input string $x_1x_2x_3x_4x_5$ can be recovered from the codeword $y_1y_2y_3y_4y_5$ assigned to it by the circuit, determine how many distinct functions f, of the type partially described above, can be defined.

$|\{0,1\}^5| = 2^5$ $\qquad$ $|\{0,1,\ldots,9,A,\ldots,F\}^5| = 16^5$

The number of functions is $(16^5)^{(2^5)}$.

$|\ |$ denotes set cardinality

(b) Select the strongest true assertion from the list below:

(i) The function f must be *invertible*.

(ii) The function f can be *invertible*.

$\boxed{\text{(iii) The function } f \text{ cannot be } \textit{invertible}.}$ because f cannot be onto.

Explain your reasoning succinctly, but clearly and convincingly.

The function f must map each element in $\{0,1\}^5$ to exactly one element in $\{0,1,\ldots,9,A,\ldots,F\}^5$. However, $|\{0,1\}^5| < |\{0,1,\ldots,9,A,\ldots,F\}^5|$, which means there are elements in $\{0,1,\ldots,9,A,\ldots,F\}^5$ that f cannot map to $\Rightarrow f$ is not onto.

(c) How many *one-to-one* functions $f : \{0,1\}^5 \to \{0,1,\ldots,9,A,B,C,D,E,F\}^5$ can be defined?

For f to be one-to-one, each input string in $\{0,1\}^5$ must map to a distinct codeword in $\{0,1,\ldots,9,A,\ldots,F\}^5$. There are two ways to count how many ways this is possible.

Method 1: Arrange the 2^5 input strings in $\{0,1\}^5$ in some order. The first one can be mapped to any one of 16^5 codewords. The second input string can be mapped to all but one of the codewords (because that one has already been assigned). The next input string can be mapped to all but two of the codewords, and so on. So the count is

$$16^5 (16^5-1)(16^5-2)\cdots(16^5-2^5+1)$$

Method 2: The answer is the number of ways of selecting 2^5 codewords out of 16^5 codewords, where the selection order matters $\Longrightarrow$ Permutations $\dfrac{16^5!}{(16^5-2^5)!}$

You may use this page for scratch work only.
Without exception, subject matter on this page will *not* be graded.

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Lab Time Lab? What lab?

Problem	Points	Your Score
Name	10	10
1	30	30
2	25	25
3	25	25
4	25	25
Total	**115**	115