Department of EECS - University of California at Berkeley

EECS 126 - Probability and Random Processes - Spring 2007

Midterm 2': 4/19/2007

SOLUTIONS

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1. (25%)

Let X, Y, Z be i.i.d. N(0, 1).

a) Show that $X+Y$ and $(X-Y)^2$ are independent;

b) Calculate $E[X+Y|X+2Y, Y-Z]$;

c) Calculate $MLE[X|X+Y, X+Z]$.

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a) We first note that $X+Y \perp X-Y$ since $\text{cov}(X+Y, X-Y) = E((X+Y)(X-Y)) = E(X^2) - E(Y^2) = 0$. Since $X+Y$ and $X-Y$ are jointly Gaussian, this implies that these random variables are independent. Consequently, $X+Y$ and $(X-Y)^2$ are independent.

b) Let $U = X+Y, V_1 = X+2Y, V_2 = X-Z$, and $\mathbf{V} = (V_1, V_2)^T$. Then

$$E[U|\mathbf{V}] = E(U\mathbf{V}^T)[E(\mathbf{V}\mathbf{V}^T)]^{-1}\mathbf{V} = [3,1]\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix}^{-1}\mathbf{V} = [3,1]\frac{1}{6}\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}\mathbf{V} = \frac{1}{6}[4,-1]\mathbf{V}.$$

Hence,

$$E[X+Y|X+2Y, Y-Z] = \frac{4}{6}(X+2Y) - \frac{1}{6}(Y-Z).$$

c) Let $W_1 = X+Y, W_2 = X+Z$, and $\mathbf{W} = (W_1, W_2)^T$. Then

$$f_{\mathbf{W}|X}[\mathbf{w}|x] = \frac{1}{2\pi}\exp\{-(w_1-x)^2/2 - (w_2-x)^2/2\}.$$

We know that $MLE[X|\mathbf{W} = \mathbf{w}] = \text{argmax}_x f_{\mathbf{W}|X}[\mathbf{w}|x]$. That is, the MLE is the minimizer of

$$g(x) = \frac{1}{2}(w_1-x)^2 + \frac{1}{2}(w_2-x)^2.$$

Writing that the derivative of $g(x)$ with respect to x is equal to zero, we find

$$(w_1-x) + (w_2-x) = 0,$$

so that $x = (w_1+w_2)/2$. Hence,

$$MLE[X|X+Y, X+Z] = \frac{1}{2}\{(X+Y) + (X+Z)\}.$$

2. (25%)

Let X, Y be independent and exponentially distributed with mean 1. Let $Z = X + 2Y$.

a) Calculate $f_{X,Z}(x, z)$;

b) Calculate $f_Z(z)$;

c) Calculate $f_{X|Z}[x|z]$;

d) Calculate $E[X|Z]$.

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a) We have

$$f_{X,Z}(x, z) = \frac{1}{2} f_{X,Y}(x, (z - x)/2) = \frac{1}{2} \exp\{-x - (z - x)/2\} = \frac{1}{2} \exp\{-(x + z)/2\} \text{ for } 0 \leq x \leq z.$$

b) We find

$$f_Z(z) = \int_0^z f_{X,Z}(x, z) dx = e^{-z/2} - e^{-z}.$$

c)

$$f_{X|Z}[x|z] = \frac{f_{X,Z}(x, z)}{f_Z(z)} = \frac{1}{2} \frac{e^{-(x+z)/2}}{e^{-z/2} - e^{-z}} = \frac{e^{-x/2}}{2(1 - e^{-z/2})} \text{ for } 0 \leq x \leq z.$$

d) We know that

$$E[X|Z = z] = \int_0^z x f_{X|Z}[x|z] dx = \frac{1}{2(1 - e^{-z/2})} \int_0^z x e^{-x/2} dx.$$

Now,

$$\int_0^z x e^{-x/2} dx = -2 \int_0^z x de^{-x/2} = -2[xe^{-x/2}]_0^z + 2 \int_0^z e^{-x/2} dx = -2ze^{-z/2} - 4[e^{-x/2}]_0^z = 4(1 - e^{-z/2}) - 2ze^{-z/2}.$$

Finally,

$$E[X|Z = z] = \frac{4(1 - e^{-z/2}) - 2ze^{-z/2}}{2(1 - e^{-z/2})} = 2 - \frac{ze^{-z/2}}{1 - e^{-z/2}}.$$

3. (25%)

Let $\mathbf{X}, \mathbf{Y}$ be random vectors defined on some common probability space.

a) Show that if they are jointly Gaussian, then $\mathbf{X} \perp \mathbf{Y}$ implies $\mathbf{X} \perp h(\mathbf{Y})$ for all function $h(.)$.

b) Show, by a counterexample, that the above fact does not hold in general if the random vectors are not jointly Gaussian.

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a) Assume that $\mathbf{X}, \mathbf{Y}$ are jointly Gaussian and $\mathbf{X} \perp \mathbf{Y}$. Then $\mathbf{X}$ and $\mathbf{Y}$ are independent. Consequently, $\mathbf{X}$ and $h(\mathbf{Y})$ are independent for any function $h(.)$. Therefore, $\mathbf{X} \perp h(\mathbf{Y})$.

b) There are many counterexamples, of course. Here is one. Let Y be $N(0,1)$, $X = Y^2$, and $h(Y) = Y^2$. Then $X \perp Y$ since $E(XY) = E(Y^3) = 0 = E(X)E(Y)$ because $E(Y) = 0$. Also, $X \not\perp h(Y)$ since $E(Xh(Y)) = E(Y^4) = 3 \neq E(X)E(h(Y)) = E(Y^2)E(Y^2) = 1$.

4. (25%)

Let X be $N(0,1)$ and $\mathbf{Z}$ be $N(0,I)$ variables in $\Re^n$, $\mathbf{v}$ a vector in $\Re^n$, and A a nonsingular matrix in $\Re^{n\times n}$.

a) Find an expression for $\sigma^2 := E((X - E[X|\mathbf{v}X + \mathbf{Z}])^2)$;

b) Calculate σ^2 for $\mathbf{v} = [\beta, 0, 0, \ldots, 0]^T$ and designate the resulting value by $g(\beta^2)$;

c) Argue, using symmetry, that for a general vector $\mathbf{v}$ one has $\sigma^2 = g(||\mathbf{v}||^2)$;

d) Show that $E((X - E[X|\mathbf{v}X + A\mathbf{Z}])^2)$ is decreasing in $||A^{-1}\mathbf{v}||^2$.

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a) We know that

$$\sigma^2 = E((X - E[X|\mathbf{Y}])^2) = 1 - \Sigma_{X,\mathbf{Y}}\Sigma_{\mathbf{Y}}^{-1}\Sigma_{\mathbf{Y},X} = 1 - \mathbf{v}^T[\mathbf{v}\mathbf{v}^T + I]^{-1}\mathbf{v}.$$

b) Let $\mathbf{v} = [\beta, 0, 0, \ldots, 0]^T$. Then

$$\sigma^2 = 1 - [\beta, 0, \ldots, 0]\text{diag}(1 + \beta^2, 1, \ldots, 1)^{-1}[\beta, 0, \ldots, 0]^T = 1 - \frac{\beta^2}{1+\beta^2} = \frac{1}{1+\beta^2} =: g(\beta^2).$$

c) Since the distribution of $\mathbf{Z}$ is invariant under rotation, we can always rotate the axes so that $\mathbf{v} = [\beta, 0, 0, \ldots, 0]^T$ where $\beta^2 = ||\mathbf{v}||^2$.

d)

Let $\mathbf{U} = \mathbf{v}X + A\mathbf{Z}$. Since A is nonsingular, observing $\mathbf{U}$ is the same as observing $\mathbf{T} := A^{-1}\mathbf{U} = A^{-1}\mathbf{v} + \mathbf{Z}$. Consequently,

$$E((X - E[X|\mathbf{U}])^2) = E((X - E[X|T])^2) = g(||A^{-1}\mathbf{v}||^2)$$

and $g(\cdot)$ is decreasing.