Department of EECS - University of California at Berkeley

EECS 126 - Probability and Random Processes - Spring 2007

Midterm 1: 2/19/2007

Solutions

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1. (10%)

You are given a 1-meter long wood stick. You choose two points A and B uniformly and independently on the stick. You cut the stick at A and at B. You are left with three pieces. What is the probability that you can form a triangle with the three pieces?

Consider the case $A < B$. The three pieces of stick have lengths $X = A, Y = B - A, Z = 1 - B$. You can make a triangle if each length is larger than the sum of the other two. That is, we need $X > Y + Z, Y > X + Z, Z > X + Y$. These inequalities in terms of A, B, C give $B \geq 1/2, A \leq 1/2, B - A \leq 1/2$.

The case $B < A$ is symmetric and gives $A \geq 1/2, B \leq 1/2, A - B \leq 1/2$.

Combining these two events, we see that they correspond to the sets of pairs (A, B) shown in the figure below:

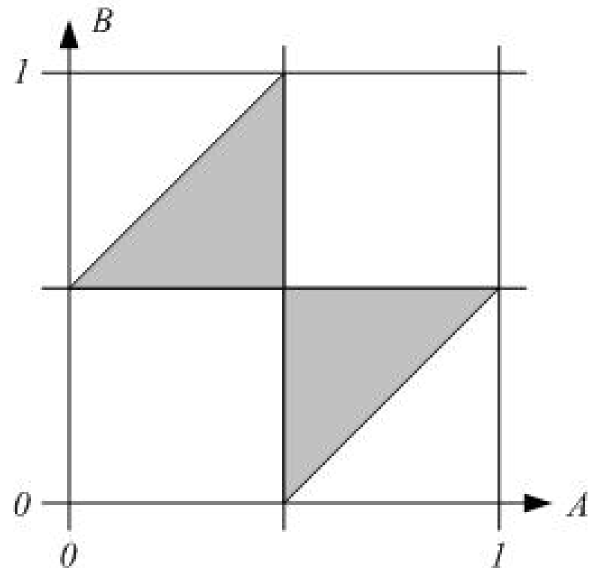

Figure 1: The shaded triangles correspond to the desired event.

The probability that (A, B) falls in the shaded set is 1/4.

2. (10%)

Let X be Poisson with parameter $\lambda > 0$. For any positive integer k, calculate

$$E(X(X-1)(X-2) \times \cdots \times (X-k)).$$

We find

$$E(X(X-1)(X-2) \times \cdots \times (X-k)) = \sum_{n \geq 0} n(n-1)(n-2) \times \cdots \times (n-k) \frac{\lambda^n}{n!} e^{-\lambda}$$
$$= \lambda^{k+1} \sum_{n \geq k+1} \frac{\lambda^{n-k-1}}{(n-k-1)!} e^{-\lambda} = \lambda^{k+1} \sum_{m \geq 0} \frac{\lambda^m}{m!} e^{-\lambda}$$
$$= \lambda^{k+1}.$$

3. (10%) Two friends agree to go to a given bar between noon and 1:00 pm and to wait for ten minutes there. Assume they choose the time they go to the bar independently and uniformly between noon and 1:00 pm, what is the probability that they meet in the bar?

Say that one friend arrives at time X and the other at time Y. They meet if $X - Y \leq 1/6$. This event corresponds to the set of pairs (X, Y) shown in the shaded set in the figure below:

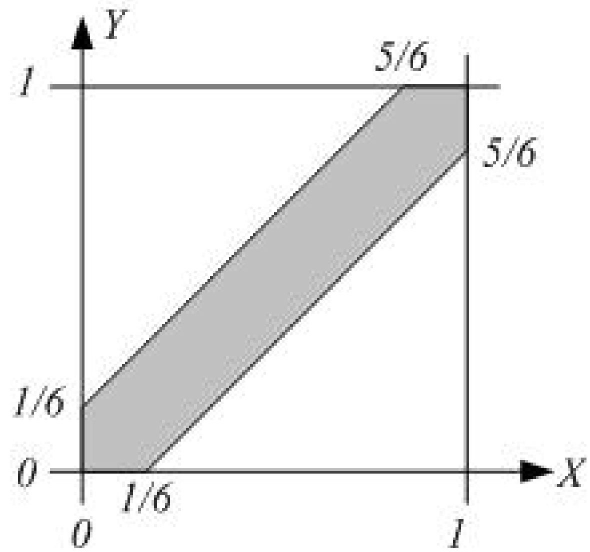

Figure 2: The shaded are corresponds to the desired event.

The probability that (X, Y) is picked in the shaded set is 11/36.

4. (10%)

You do not feel too well and you wonder why. The prior probability that you have the flu, some food poisoning, or some other disease D is 10%, 5%, and 15%, respectively. The probability that you feel this sick if you have the flu, food poisoning. or the disease D, is 80%, 95%, 20%, respectively. What is the probability that you are sick because of food poisoning?

Let B be the event "you feel sick," A_1 the event "you have the flu," A_2 the event "you have some food poisoning," and A_3 the event "you have some other disease."

We are given $P(A_1) = 0.1, P(A_2) = 0.05, P(A_3) = 0.15$ and $P[B|A_1] = 0.8, P[B|A_2] = 0.95, P[B|A_3] = 0.2$. We find the probability $P[A_2|B]$ by using Bayes' rule:

$$
\begin{aligned}
P[A_2|B] &= \frac{P(A_2)P[B|A_2]}{P(A_1)P[B|A_1] + P(A_2)P[B|A_2] + P(A_3)P[B|A_3]} \\
&= \frac{0.05 \times 0.95}{0.1 \times 0.8 + 0.05 \times 0.95 + 0.15 \times 0.2} \\
&\approx 0.30.
\end{aligned}
$$

5. (10%)

Can you find a probability space and events A, B so that

$$P[A|B] > P(A) \text{ and } P[B|A] < P(B)?$$

The answer is no. Indeed, the first inequality implies

$$P(A \cap B) > P(A)P(B)$$

while the second implies

$$P(A \cap B) < P(A)P(B),$$

and these are clearly incompatible.

6. (10%)

How many times, on average, do you have to roll a balanced die until you see all six faces at least once?

You recall that if a coin flip has a probability p of producing a head, you have to flip it $1/p$ times, on average to get the first head.

You roll the die once to get the first face. The probability of getting a different face after that roll is $5/6$, so that it takes $6/5$ rolls, on average, to get a second face. After that roll, the probability of getting a face different from the first two is $4/6$, so that it takes $6/4$ rolls to get a third face, and so on.

Hence, the answer is

$$1 + 6/5 + 6/4 + 6/3 + 6/2 + 6/1 \approx 14.7.$$

7. (10%)

Let X be a random variable that is uniform in $[0, 1]$. Calculate the variance of X^n for $n \geq 1$.

We find

$$\mathrm{var}(X^n) = E(X^{2n}) - E(X^n)^2 = \frac{1}{2n+1} - \frac{1}{(n+1)^2}.$$

We used the fact that

$$E(X^m) = \int_0^1 x^m dx = \frac{1}{m+1}.$$

8. (10%)

State and prove Markov's inequality.

Markov's inequality states that if X is a nonnegative random variable, then

$$P(X \geq a) \leq \frac{E(X)}{a}.$$

To prove the inequality, we observe that

$$X \geq a1\{X \geq a\}.$$

Taking the expectation, we get $E(X) \geq aP(X \geq a)$.

9. (10%)

You throw a dart randomly and uniformly in a unit circle. Let X be the distance between the dart and the center. Calculate $E(sin(X))$.

The pdf of X is $f(x) = 2x$ for $0 \le x \le 1$ and $f(x) = 0$ for x not in $[0,1]$. Consequently,

$$E(\sin(X)) = \int_0^1 \sin(x) 2x dx = -\int_0^1 2x d\cos(x) = -[2x\cos(x)]_0^1 + 2\int_0^1 \cos(x) dx = -2\cos(1) + 2\sin(1).$$

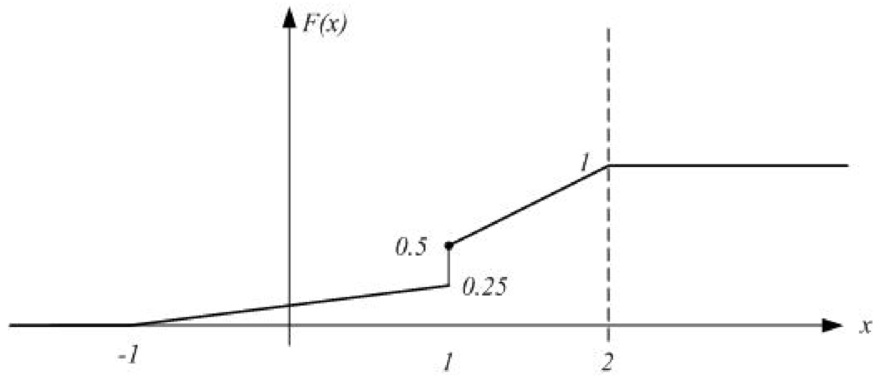

Figure 3: The cpdf of X.

10. (10%)

The random variable X has the c.p.d.f. shown above. Calculate $E(X)$ and $\text{var}(X)$.

We see that

$$E(X) = \int_{-1}^{1} x \times 0.125dx + 1 \times 0.25 + \int_{1}^{2} x \times 0.5dx = 0 + 0.25 + 0.75 = 1.$$

Also,

$$E(X^2) = \int_{-1}^{1} x^2 \times 0.125dx + 1 \times 0.25 + \int_{1}^{2} x^2 \times 0.5dx = 0.25/3 + 0.25 + (2^3 - 1^3)0.5/3 = 1.5.$$

Hence,

$$\text{var}(X) = 1.5 - 1^2 = 0.5.$$