EE 120: Signals and Systems
Department of EECS
UC BERKELEY

MIDTERM 2
12 November 2008

LAST Name Transformer FIRST Name Z

Discussion Time Randomly Chosen.

- **(10 Points)** Print your name and discussion time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 90 minutes to complete. You will be given at least 90 minutes, up to a maximum of 110 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except two double-sided $8.5" \times 11"$ sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 8.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the eight numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

You may use this page for scratch work only.
Without exception, subject matter on this page will *not* be graded.

MT2.1 (31 Points) Consider a discrete-time LTI filter Q whose impulse response values are

$$\forall n \in \mathbb{Z}, \quad q(n) = \alpha^n \cos(\omega_0\, n)\, u(n),$$

where u is the unit-step, ω_0 is real, and α is nonzero (but possibly complex).

(a) **(15 Points)** Show that the transfer function of the filter is

$$\widehat{Q}(z) = \frac{z^2 - \alpha\, \cos(\omega_0)\, z}{z^2 - 2\,\alpha\, \cos(\omega_0)\, z + \alpha^2},$$

and determine the corresponding region of convergence RoC(q).

$$\hat{Q}(z) = \sum_{n=0}^{\infty} \alpha^n \cos(\omega_0 n) z^{-n} = \frac{1}{2}\sum_{n=0}^{\infty} \alpha^n e^{i\omega_0 n} z^{-n} + \frac{1}{2}\sum_{n=0}^{\infty} \alpha^n e^{-i\omega_0 n} z^{-n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} (\alpha e^{i\omega_0} z^{-1})^n + \frac{1}{2}\sum_{n=0}^{\infty} (\alpha e^{-i\omega_0} z^{-1})^n = \frac{1}{2}\left(\frac{1}{1-\alpha e^{i\omega_0} z^{-1}} + \frac{1}{1-\alpha e^{-i\omega_0} z^{-1}}\right)$$

$$= \frac{1}{2}\left(\frac{z}{z-\alpha e^{i\omega_0}} + \frac{z}{z-\alpha e^{-i\omega_0}}\right) = \frac{z}{2}\,\frac{z-\alpha e^{-i\omega_0} + z - \alpha e^{i\omega_0}}{(z-\alpha e^{i\omega_0})(z-\alpha e^{-i\omega_0})}$$

$$= \frac{\frac{1}{2} z\,[2z - \alpha(e^{i\omega_0} + e^{-i\omega_0})]}{z^2 - \alpha(e^{i\omega_0} + e^{-i\omega_0}) z + \alpha^2 e^{i\omega_0} e^{-i\omega_0}} = \frac{z^2 - \alpha \cos\omega_0}{z^2 - 2\alpha\cos\omega_0\, z + \alpha^2}$$

RoC: $|z| > |\alpha|$

(b) **(8 Points)** Without much work, determine the set of all value(s) of α for which the filter Q is BIBO stable? Use succinct, yet clear and convincing, reasoning.

Q is BIBO stable if, and only if, $\sum |q(n)| < \infty$.

$$\sum_n |q(n)| = \sum_n |\alpha^n \cos\omega_0 n\, u(n)| = \sum_n |\alpha^n|\,|\cos\omega_0 n|\,|u(n)| \leq \sum_n |\alpha|^n$$

So Q is BIBO stable for all α such that $|\alpha| < 1$. We can also see this from the RoC in (a), which must include the unit circle for BIBO stability.

(c) **(8 Points)** Let x be the input and y the output of the filter Q. Determine the linear, constant-coefficient difference equation that governs the input-output behavior of the filter.

$$\hat{Q}(z) = \frac{\hat{Y}(z)}{\hat{X}(z)} = \frac{1 - \alpha\cos\omega_0\, z^{-1}}{1 - 2\alpha\cos\omega_0\, z^{-1} + \alpha^2 z^{-2}} \Rightarrow (1 - 2\alpha\cos\omega_0\, z^{-1} + \alpha^2 z^{-2})\hat{Y}(z) = (1 - \alpha\cos\omega_0\, z^{-1})\hat{X}(z)$$

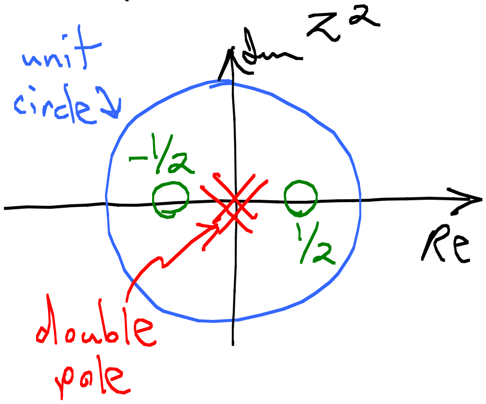
$$\Rightarrow y(n) - 2\alpha\cos\omega_0\, y(n-1) + \alpha^2 y(n-2) = x(n) - \alpha\cos\omega_0\, x(n-1)$$

MT2.2 (64 Points) Consider a discrete-time FIR filter G whose transfer function is

$$\widehat{G}(z) = 1 - \frac{1}{4}z^{-2}.$$

(a) **(8 Points)** Draw the pole-zero diagram for $\widehat{G}$ and determine RoC(g), its corresponding region of convergence.

$\hat{G}(z) = \dfrac{z^2 - 1/4}{z^2} \Rightarrow$ zeros: $1/2, -1/2$; poles: $0, 0$ (double pole @ the origin)



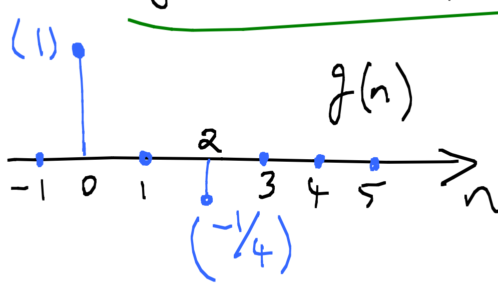
RoC(g): $|z| > 0$

Everywhere except @ $z = 0$.

Even $|z| = \infty$ is included in the RoC.

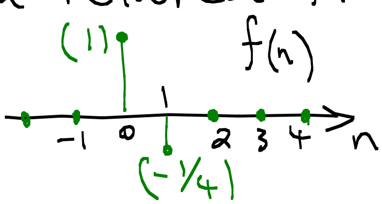
(b) **(8 Points)** Determine, and provide a well-labeled plot of, $g(n)$, the filter's impulse response.

$\hat{G}(z) = 1 - \frac{1}{4}z^{-2} \Rightarrow g(n) = \delta(n) - \frac{1}{4}\delta(n-2)$



(c) **(10 Points)** Provide a well-labeled plot of $|G(\omega)|$, the filter's magnitude response.

Note that $g(n)$ can be thought of as the upsampled version of a related filter whose impulse response is f, as follows:



$f(n) \longrightarrow \boxed{\uparrow 2} \longrightarrow g(n) = \begin{cases} f\left(\frac{n}{2}\right) & n \bmod 2 = 0 \\ 0 & \text{elsewhere.} \end{cases}$

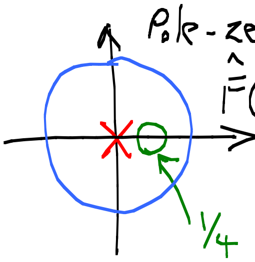
Equivalently $\hat{G}(z) = \hat{F}(z^2)$, where $\hat{F}(z) = 1 - \frac{1}{4}z^{-1}$;

or, in terms of frequency responses, $G(\omega) = F(2\omega)$

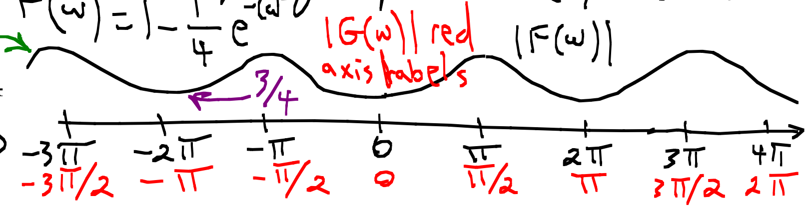
where $F(\omega) = 1 - \frac{1}{4}e^{-i\omega}$

Pole-zero diagram for $\hat{F}$

$\hat{F}(z) = \dfrac{z - 1/4}{z}$



This is the high-pass counterpart of $\dfrac{1}{1 - 1/4 z^{-1}}$ $\Rightarrow$



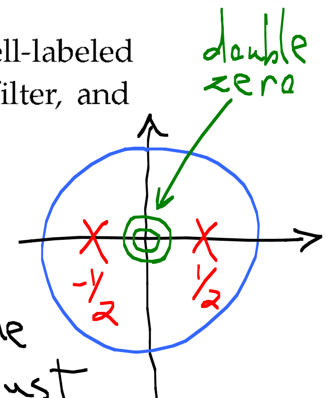
(d) **(38 Points)** Suppose we want to design a *stable and causal inverse* for the filter G. If G represents a distortion that we want to remove from a desired signal, the inverse system enables us to recover the original signal from the corrupted version. We want the inverse filter to be causal, so we can remove the distortion in real time.

The inverse filter is one which, if placed in cascade with G, will produce an overall transfer function equal to unity. That is, if the inverse filter is H, then

$$\widehat{G}(z)\widehat{H}(z) = 1.$$

(i) **(9 Points)** Provide a reasonably-simple expression for, and a well-labeled pole-zero plot of, the transfer function $\widehat{H}(z)$ of the inverse filter, and specify its region of convergence.

$$\hat{G}(z) = 1 - \tfrac{1}{4}z^{-2} \Rightarrow \hat{H}(z) = \frac{1}{1-\frac{1}{4}z^{-2}} = \frac{z^2}{z^2 - 1/4} = \frac{z^2}{(z-1/2)(z+1/2)}$$



zeros: 0, 0
poles: $-1/2$, $+1/2$

We want H to be stable and causal, so RoC(h) must be $|z| > 1/2$ (Outside outermost pole (incl. ∞), and includes the unit circle.

$$\hat{H}(z) = \frac{1}{1-\frac{1}{4}z^{-2}}$$

(ii) **(9 Points)** Explain what it is about the transfer function $\widehat{G}$ that guarantees the existence of a causal and stable inverse.

The poles of G are the zeros of H, and the zeros of G are the poles of H. The filter G is causal & stable, so all its poles are inside the unit circle. But for its inverse H to be causal & stable, the poles of H (which are the zeros of G) must also be inside the unit circle. So, what is special here is that all the poles AND zeros of G are inside the unit circle. This guarantees the existence of a causal & stable inverse.

(iii) **(9 Points)** Without doing much work, determine numerical values for

$$\sum_{n=-\infty}^{\infty} h(n) \qquad \text{and} \qquad h(0).$$

$$\hat{H}(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \Rightarrow \hat{H}(1) = \sum_{n=-\infty}^{\infty} h(n)$$

We know we can evaluate $\hat{H}$ at $z=1$ b/c the unit circle is included in RoC (h).

$$\hat{H}(1) = \left.\frac{1}{1-\frac{1}{4}z^{-2}}\right|_{z=1} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3} \Rightarrow \sum_n h(n) = \frac{4}{3}$$

H is causal $\Rightarrow$ Apply initial value theorem:

$$h(0) = \lim_{z\to\infty} \hat{H}(z) = \lim_{z\to\infty} \frac{z^2}{z^2 - 1/4} = 1 \Rightarrow h(0) = 1$$

(iv) **(11 Points)** Determine an explicit expression for, and provide a well-labeled plot of, $h(n)$, the impulse response values of the inverse filter H.

Method 1: Use the transform pair from problem 1:

$$\alpha^n \cos(\omega_0 n) u(n) \overset{Z}{\longleftrightarrow} \frac{z^2 - \alpha \cos\omega_0}{z^2 - 2\alpha\cos\omega_0 z + \alpha^2}$$

$$\hat{H}(z) = \frac{z^2}{z^2 - \frac{1}{4}} \Rightarrow \alpha = \frac{i}{2}, \quad \omega_0 = \frac{\pi}{2} \Rightarrow$$

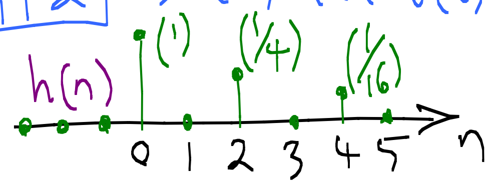
$$h(n) = \left(\frac{i}{2}\right)^n \cos\left(\frac{\pi}{2}n\right) u(n) = \left(\frac{1}{2}\right)^n i^n \cos\left(\frac{\pi}{2}n\right) u(n) = \left(\frac{1}{2}\right)^n e^{i\frac{\pi}{2}n} \frac{e^{i\frac{\pi}{2}n} + e^{-i\frac{\pi}{2}n}}{2} u(n)$$

$$h(n) = \left(\frac{1}{2}\right)^n \frac{e^{i\pi n} + 1}{2} u(n) \Rightarrow h(n) = \left(\frac{1}{2}\right)^n \frac{(-1)^n + 1}{2} u(n)$$

Note: $h(n) = 0$ $\forall n$ odd

Method 2: Recognize that $h(n)$ is the factor-2 upsampled version of $v(n)$, where $v(n) \rightarrow \boxed{\uparrow 2} \rightarrow h(n)$ and $\hat{V}(z) = \frac{1}{1-\frac{1}{4}z^{-1}}$

Clearly, $v(n) = \left(\frac{1}{4}\right)^n u(n)$, so $h(n)$ is



MT2.3 (10 Points) True or false? True

$$\delta(at - T) = \frac{1}{|a|}\delta(t - T/a),$$

where δ is the Dirac delta, a is a nonzero real quantity, and T is real. Explain your reasoning. Two Dirac deltas are "equal" if the behave the same way.

First we show that they have the same areas.

If $a>0$, $\int_{-\infty}^{\infty} \delta(at-T)\,dt = \frac{1}{a}\int_{-\infty}^{\infty} \delta(\tau)\,\frac{d\tau}{a} = \frac{1}{a} = \frac{1}{|a|}$

$\tau = at - T \Rightarrow d\tau = a\,dt$

If $a<0$, $\int_{-\infty}^{\infty} \delta(at-T)\,dt = \frac{1}{a}\int_{+\infty}^{-\infty} \delta(\tau)\,d\tau = \frac{-1}{a}\int_{-\infty}^{+\infty} \delta(\tau)\,d\tau = \frac{1}{|a|}$

$\Rightarrow \int_{-\infty}^{\infty} \delta(at-T)\,dt = \frac{1}{|a|}$

$\int_{-\infty}^{\infty} \delta(t-\frac{T}{a})\,dt = 1 \Rightarrow \int_{-\infty}^{\infty} \frac{1}{|a|}\delta(t-aT)\,dt = \frac{1}{|a|} \Longrightarrow \delta(at-T)$ & $\frac{1}{|a|}\delta(t-T/a)$ have equal areas.

They also have identical sifting properties:

$\int \delta(at-T)\,x(t)\,dt = \int_{-\infty}^{\infty} \delta(\lambda-T)\,x(\frac{\lambda}{a})\frac{d\lambda}{a} = \frac{x(T/a)}{a}$ $a>0$

$\lambda = at \Rightarrow d\lambda = a\,dt$

or $= x(T/a)/{-a}$ $a<0$ $\underbrace{\qquad}_{=\frac{1}{|a|}x(\frac{T}{a})}$

$\int_{-\infty}^{\infty} \frac{1}{|a|}\delta(t-\frac{T}{a})\,x(t)\,dt = \frac{1}{|a|}\int \delta(t-\frac{T}{a})\,x(t)\,dt = \frac{x(T/a)}{|a|}$

You may use the blank space below for scratch work. Nothing written below this line on this page will be considered in evaluating your work.

$\lambda = at \Rightarrow d\lambda = a\,dt$ $\int \delta(\lambda-T)\,x(\frac{\lambda}{a})\frac{d\lambda}{a} = \frac{x(\frac{T}{a})}{a}$

$\frac{1}{|a|}\int \delta(t-\frac{T}{a})\,x(t)\,dt = \frac{1}{|a|}x(\frac{T}{a})$

LAST Name Transformer FIRST Name Z

Discussion Time Randomly Chosen

Problem	Points	Your Score
Name	10	10
1	31	31
2	64	64
3	10	10
Total	**115**	115

Yay!