Problem 1 *(Short questions.)* 35 Points (5 Points each)

(a) For the following system, with input $x[n]$ and output $y[n]$, circle whether the statements are true or false.

$$y[n] = \begin{cases} x[n] & n \geq 1 \\ 0 & n = 0 \\ x[n+1] & n \leq -1 \end{cases}$$

T F the system is linear

T F the system is time-invariant

T F the system is memoryless

T F the system is stable

T F the system is causal

(b) A discrete-time LTI system has the impulse response

$$h[n] = 3^n \cdot u[4-n]$$

Circle whether the system is stable or unstable. Give a short justification of your answer in the additional box.

The system is: stable

unstable

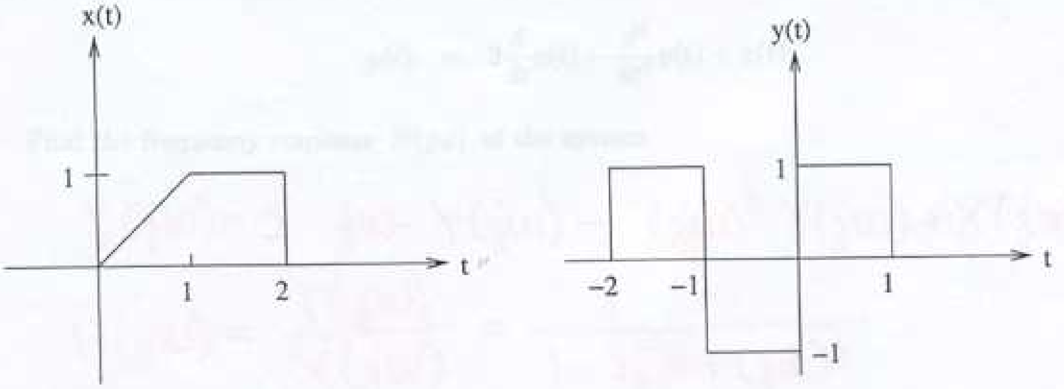
For the system to be BIBO stable, need

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

In this case,

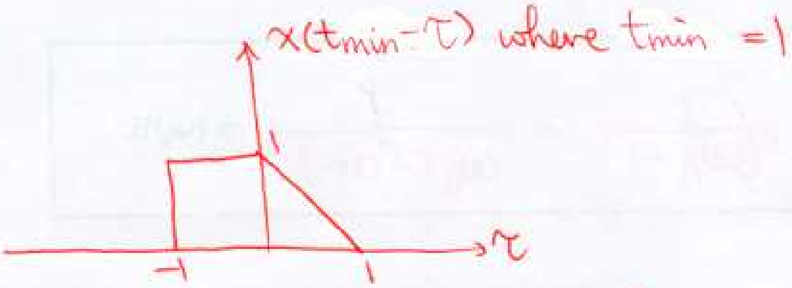
$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{4} 3^n = \frac{3^4}{1-\frac{1}{3}} < \infty$$

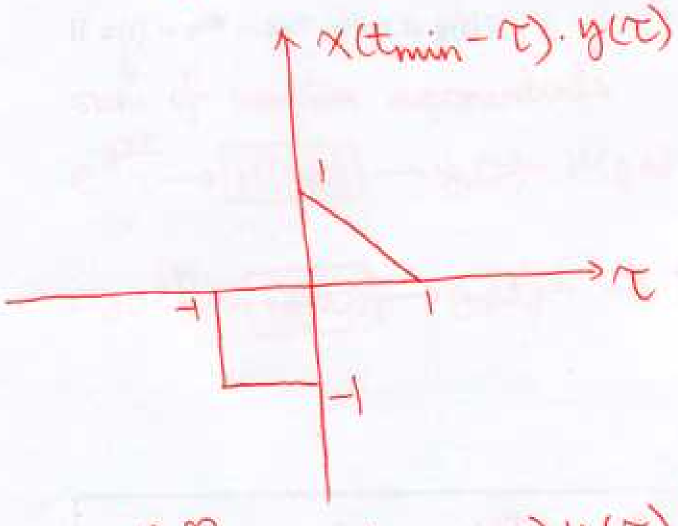
(c) The signals $x(t)$ and $y(t)$ are shown in the following figure.



Let $z(t) = x(t) * y(t)$. Find the value of t where $z(t)$ is minimum, and the value of $z(t)$ at that point.

$t = 1$	$z_{min} = -\frac{1}{2}$





$$\therefore \int_{\tau=-\infty}^{\infty} x(t_{min}-\tau)\,y(\tau)\,d\tau = -\frac{1}{2}$$

(d) A continuous time LTI system with input $x(t)$ and output $y(t)$ is described by the following constant coefficient differential equation:

$$y(t) = 2\frac{d}{dt}y(t) - \frac{d^2}{dt^2}y(t) + x(t)$$

Find the frequency response $H(j\omega)$ of the system.

$$Y(j\omega) = 2 \cdot j\omega \cdot Y(j\omega) - (j\omega)^2 Y(j\omega) + X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{1 - 2j\omega + (j\omega)^2}$$

$$= \frac{1}{1 - \omega^2 - 2j\omega} = \frac{1}{(1 - j\omega)^2}$$

$H(j\omega) = \frac{1}{1-\omega^2-2j\omega} = \frac{1}{(1-j\omega)^2}$

If $x(t) = e^{j2t} - 3e^{jt}$, what is $y(t)$?

sum of complex exponentials

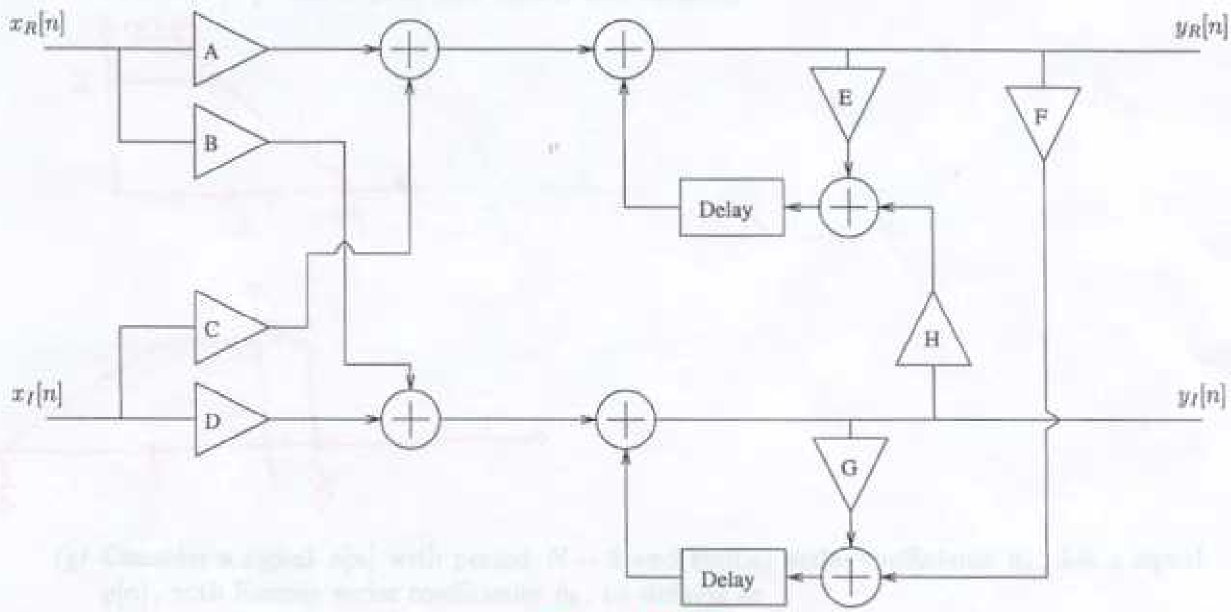
$$e^{j2t} \rightarrow \boxed{H(j\omega)} \rightarrow y_1(t) = H(j2)e^{j2t} = \frac{e^{j2t}}{(1-2j)^2} = \frac{e^{j2t}}{-3-4j}$$

$$-3e^{jt} \rightarrow \boxed{H(j\omega)} \rightarrow y_2(t) = -3 \cdot H(j)e^{jt} = \frac{-3e^{jt}}{(1-j)^2} = \frac{3e^{jt}}{2j}$$

$y(t) = -\frac{1}{3+4j}e^{j2t} + \frac{3}{2j}e^{jt}$

(e) Find the correct real gains in the block diagram below so that the input and output are related by the complex difference equation:

$$y[n] - 2e^{j3\pi/4}y[n-1] = (3j)x[n]$$



$y[n] = 3j \cdot x[n] + 2\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}j\right) y[n-1]$

$y_R[n] = -3 \cdot x_I[n] - \sqrt{2}\, y_R[n-1] - \sqrt{2}\, y_I[n-1]$

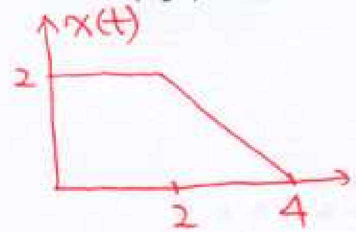
$y_I[n] = 3x_R[n] - \sqrt{2}\, y_I[n-1] + \sqrt{2}\, y_R[n-1]$

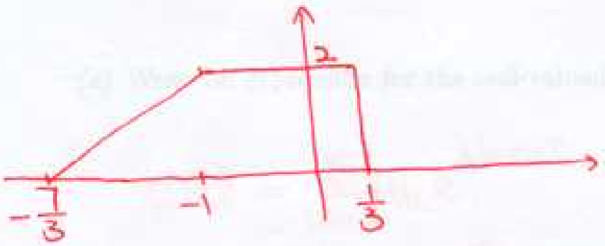
$A =$ 0	$B =$ 3	$C =$ -3
$D =$ 0	$E =$ $-\sqrt{2}$	$F =$ $\sqrt{2}$
$G =$ $-\sqrt{2}$	$H =$ $-\sqrt{2}$	

(f)

$$\text{Given} \quad x(t) = \begin{cases} 2, & 0 \le t < 2, \\ 4-t, & 2 \le t < 4, \\ 0, & \text{otherwise.} \end{cases}$$

Plot $x\left(\frac{1-3t}{2}\right)$. Label your axes clearly and carefully!





(g) Consider a signal $x[n]$ with period $N = 8$ and Fourier series coefficients a_k. Let a signal $y[n]$, with Fourier series coefficients b_k, be defined as

$$y[n] = x[n] \cos\left(\frac{\pi n}{2}\right)$$

Find b_2 and b_5 in terms of the a_k

$$y[n] = x[n] \cdot \frac{1}{2}\left(e^{j\pi\frac{n}{2}} + e^{-j\pi\frac{n}{2}}\right)$$

$$b_k = \frac{1}{8}\sum_{n=0}^{7} y[n] e^{-j\frac{2\pi}{8}kn}$$

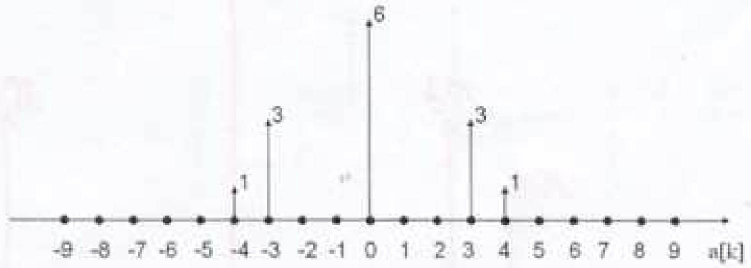
$$= \frac{1}{8}\sum_{n=0}^{7} x[n] \frac{1}{2}\left[e^{-j\frac{2\pi}{8}(k-2)n} + e^{-j\frac{2\pi}{8}(k+2)n}\right]$$

$$= \frac{1}{2} a_{\langle k-2\rangle_8} + \frac{1}{2} a_{\langle k+2\rangle_8}$$

$b_2 = \frac{1}{2}a_0 + \frac{1}{2}a_4$	$b_5 = \frac{1}{2}a_3 + \frac{1}{2}a_7$

Problem 2 **15 Points**

$x(t)$ is a real-valued periodic signal, with period $T = 10$. The Fourier series coefficients of $x(t)$ are also real-valued, and are shown below



(a) Write an expression for the real-valued signal $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \left(e^{j(-4)\frac{2\pi}{10}t} + e^{j(4)\frac{2\pi}{10}t}\right) +$$

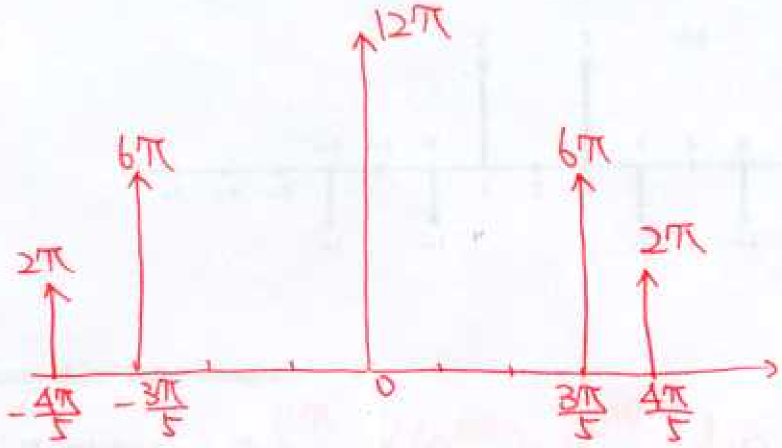
$$3\left(e^{j(-3)\frac{2\pi}{10}t} + e^{j(3)\frac{2\pi}{10}t}\right) +$$

$$6\, e^{j\cdot 0\cdot\frac{2\pi}{10}t}$$

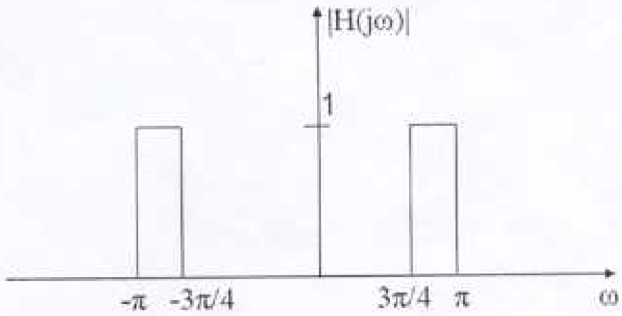
$$= 2\cos\left(\frac{4\pi}{5}t\right) + 6\cos\left(\frac{3\pi}{5}t\right) + 6$$

$$\boxed{x(t) = 6 + 6\cos\left(\frac{3\pi}{5}t\right) + 2\cos\left(\frac{4\pi}{5}t\right)}$$

(b) Plot the CTFT $X(j\omega)$ of the signal $x(t)$. Clearly label both axes of the plot.



(c) The signal $x(t)$ is now the input to an LTI system, whose frequency response $H(j\omega)$ is given by $\angle H(j\omega) = -2\omega$ and $|H(j\omega)|$ shown in the following figure.



Write an expression for the output of the LTI system, $y(t)$

Only the impulses at $\omega = \pm\frac{4\pi}{5}$ are reserved

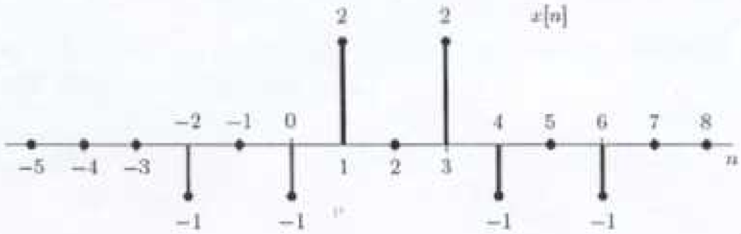
$\angle H(j\omega) = -2\omega$ results in a shift in time by 2.

$(H(j\omega) = e^{-j2\omega})$

$y(t) = 2\cos\left(\frac{4\pi}{5}(t-2)\right)$

Problem 3 25 Points

Let $X(e^{j\omega})$ denote the discrete-time Fourier transform of the signal $x[n]$ shown below.



Evaluate these three integrals:

(a) $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$ $= \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega \cdot 0} \, d\omega = 2\pi \cdot x[0]$

$\int_{-\pi}^{\pi} X(e^{j\omega})d\omega =$ -2π

(b) $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$ $= 2\pi \sum |x[n]|^2$

$\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega =$ 24π

(c) $\int_{-\pi}^{\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 d\omega$ $= 2\pi \sum |n \cdot x[n]|^2$

$= 2\pi \left([(-2)(-1)]^2 + [1 \cdot 2]^2 + (2 \cdot 3)^2 + (4 \cdot (-1))^2 + (6 \cdot (-1))^2 \right)$

$= 2\pi (4 + 4 + 36 + 16 + 36)$

$= 192\pi$

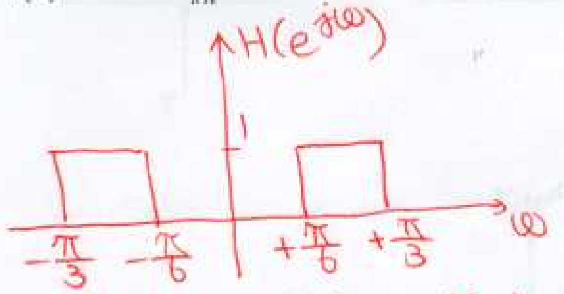
$\int_{-\pi}^{\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 d\omega =$ 192π

Now, consider the signal

$$x[n] = \sin\left(\frac{\pi n}{4}\right) - 2\cos\left(\frac{\pi n}{2}\right)$$

Suppose that this signal is the input to LTI systems with the following impulse responses. Determine the output $y[n]$ in each case.

(d) $h[n] = \frac{\sin(\pi n/3) - \sin(\pi n/6)}{\pi n}$

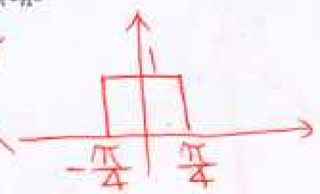


The $\sin\left(\frac{\pi n}{4}\right)$ will be reserved entirely while the $\cos\left(\frac{\pi n}{2}\right)$ will be cut off.

$y[n] = \sin\frac{\pi n}{4}$

(e) $h[n] = \frac{\sin(\pi n/4)\sin(\pi n/2)}{\pi^2 n^2}$

$H(e^{j\omega}) = \frac{1}{2\pi} \cdot \Big($



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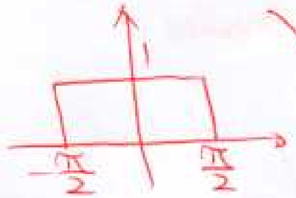
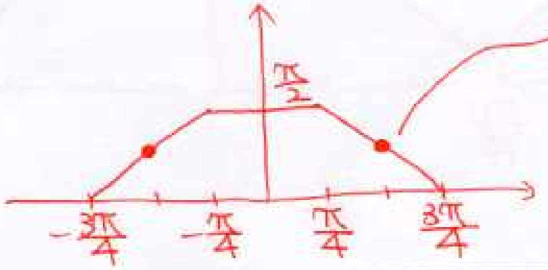


$\Big)$

$= \frac{1}{2\pi} \cdot$

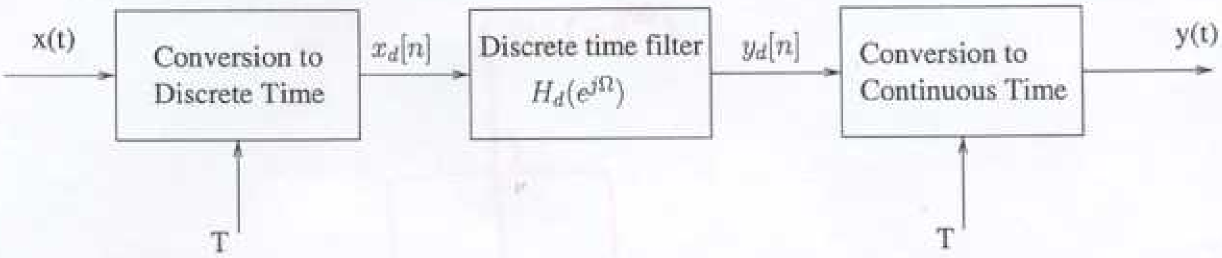


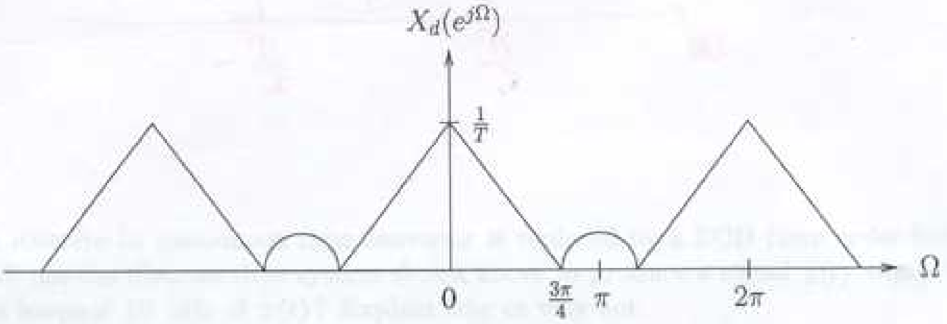
The $\cos\left(\frac{\pi n}{2}\right)$ term will take an additional $\frac{1}{2}$ factor

$y[n] = \frac{1}{4}\sin\left(\frac{\pi n}{4}\right) - \frac{1}{4}\cos\left(\frac{\pi n}{2}\right)$

Problem 4 25 Points

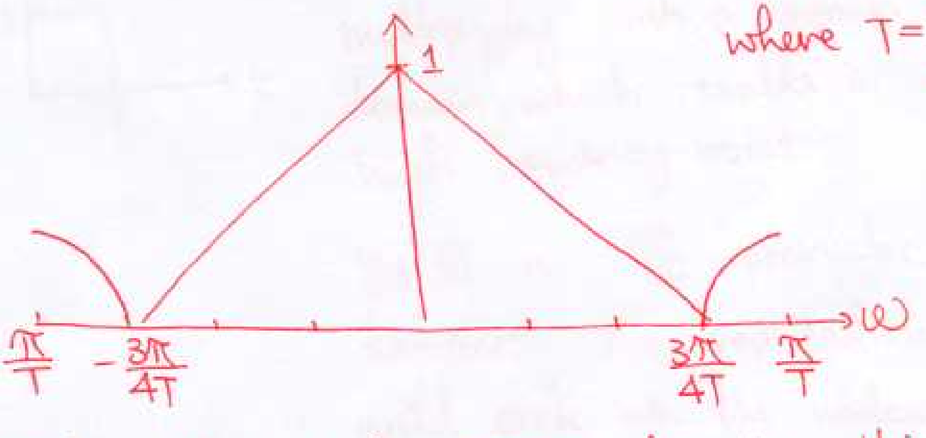
An audio signal $x(t)$, bandlimited to 20 kHz, is sampled at its Nyquist rate to produce the discrete time signal $x_d[n]$ with spectrum shown below.





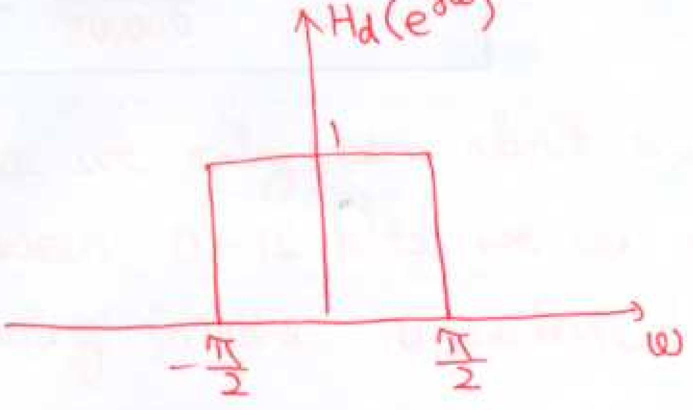
(a) Plot the spectrum of the continuous time signal $x(t)$ (i.e., plot $X(j\omega)$, the CTFT of $x(t)$). Assume the continuous to discrete time conversion is ideal.

where $T = \frac{1}{40000}$



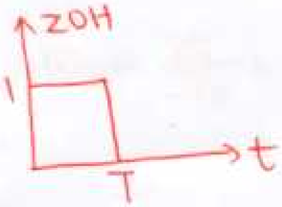
Note: This is in CTFT domain, this is not 2π periodic.

(b) We want to filter $x(t)$ to keep only the lowpass component between 0 and 10 kHz, i.e., we want $y(t)$ to represent the lowpass component of $x(t)$. Plot the spectrum of the filter $H_d(e^{j\Omega})$ needed to accomplish this task. Assume that the discrete to continuous time conversion is ideal.



(c) If the discrete to continuous time converter is replaced by a ZOH (zero order hold), can we still use the discrete time system shown above to produce a signal $y(t)$ which is equal to the lowpass 10 kHz of $x(t)$? Explain why or why not.

No.



Convolving with a square corresponds to multiplying with a sinc in frequency domain, which results in undesirable high frequency noise

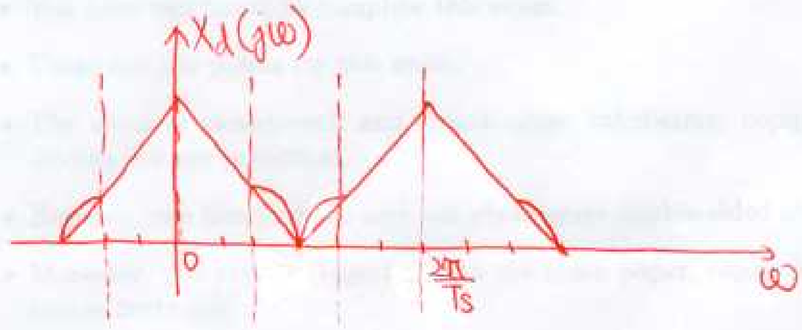
$y_d(t)$ is $\frac{2\pi}{T}$ - preriodic in CTFT domain. The replications at $\frac{2\pi k}{T}$ will pick up the undesirable high freq content of the ~~[illegible]~~ sinc function.

(You can look at HW #7 Prob 4)

(d) What is the maximum value of T in the system shown above, such that $y(t)$ can equal the lowpass 10 kHz of $x(t)$ with a suitably chosen discrete time filter?

$$T = \frac{1}{30{,}000}$$

Since we only care about the content between 0–10 kHz, we can allow aliasing in the 10–20 kHz band



$$\omega_s = \frac{2\pi}{T_s} = 30{,}000\ \text{Hz} \cdot 2\pi$$