EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 2
Department of Electrical Engineering and Computer Sciences 13 November 2008
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name Gauss FIRST Name Ian

Lab Time ?

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except two double-sided $8.5" \times 11"$ sheets of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 10.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the ten numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

MT2.1 (55 Points) Consider a discrete-time LTI filter F whose frequency response is given by

$$\forall \omega \in \mathbb{R}, \quad F(\omega) = 1 - \frac{1}{4} e^{i\omega}.$$

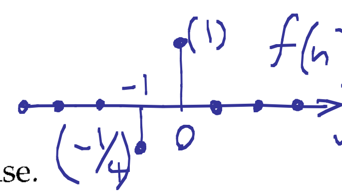
Recall that $F(\omega) = \sum_{n=-\infty}^{\infty} f(n)\, e^{-i\omega n}$.

(a) Determine an expression, as well as a well-labeled stem plot, for the filter's impulse response $f(n)$. Is the filter causal?

Only two terms in the expansion $F(\omega) = \sum_n f(n) e^{-i\omega n}$ are non-zero: $n=0$ and $n=-1$. That is, $F(\omega) = f(-1) e^{i\omega} + f(0)$

$\Longrightarrow f(0) = 1$, $f(-1) = -\frac{1}{4}$, and $f(n) = 0$ elsewhere.

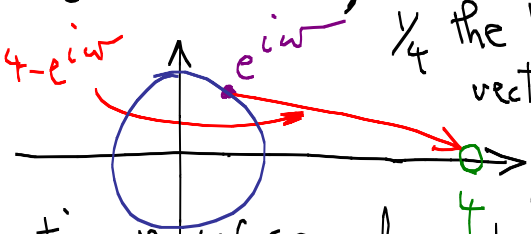
$f(n) = \delta(n) - \frac{1}{4}\delta(n+1)$



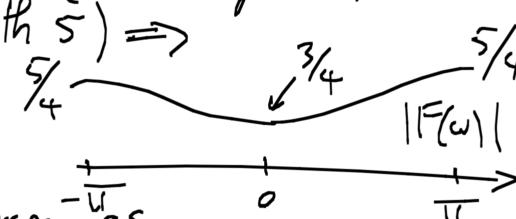
(b) In this part you explore $|F(\omega)|$, the filter's magnitude response.

(i) Provide a well-labeled plot of $|F(\omega)|$. Be sure to explain how you get your plot. (We recommend the graphical method taught in lecture).

$F(\omega) = \frac{1}{4}(4 - e^{i\omega}) \Longrightarrow |F(\omega)| = \frac{1}{4}|4 - e^{i\omega}|$. Using a graphical method, we note that $4 - e^{i\omega}$ is the vector joining $e^{i\omega}$ (which is on the unit circle) and the point 4 on the real axis. $|F(\omega)|$ is simply $\frac{1}{4}$ the length of that vector. It's easy to see that the vector is shortest for $\omega = 0$ (with length 3) and longest for $\omega = \pm\pi$ (w/ length 5) $\Rightarrow$



Curvature obtained by noting that for equal angular changes in ω near 0 & $\pm\pi$, the length of the vector doesn't change as much as for other ranges.



(ii) Specify whether the filter is low-pass, band-pass, high-pass, or none of these types. Explain.

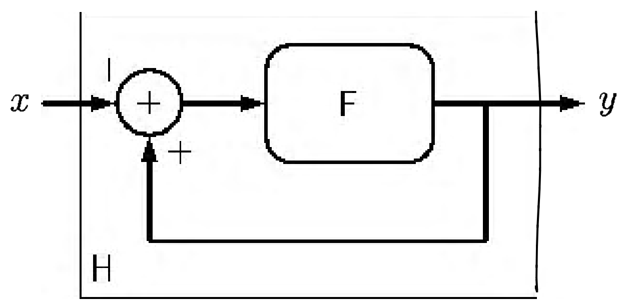
Filter in low-pass b/c it favors higher frequencies (near $\pm\pi$) over lowe frequencies (near 0).

(iii) What is the response of the filter to the input signal $1 + (-1)^n$? Explain.

Input $x(n) = 1 + (-1)^n = e^{i0n} + e^{i\pi n} \Rightarrow$ The output is

$y(n) = F(0) e^{i0n} + F(\pi) e^{i\pi n} = \frac{3}{4} + \frac{5}{4}(-1)^n$

(c) Suppose we create a composite system H by placing the filter F in the feedback configuration shown below:



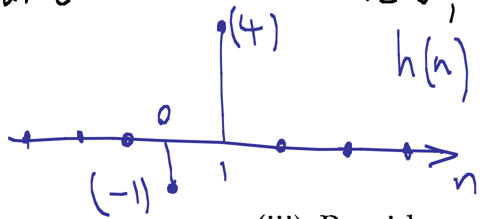
(i) Determine the frequency response $H(\omega)$ of the composite system.

$$H(\omega) = \frac{\text{Forward Gain}}{1 - \text{Loop Gain}} = \frac{F(\omega)}{1 - F(\omega)} = \frac{1 - \frac{1}{4}e^{i\omega}}{1 - (1 - \frac{1}{4}e^{i\omega})}$$

$$H(\omega) = \frac{1 - \frac{1}{4}e^{i\omega}}{\frac{1}{4}e^{i\omega}} = 4e^{-i\omega} - 1$$

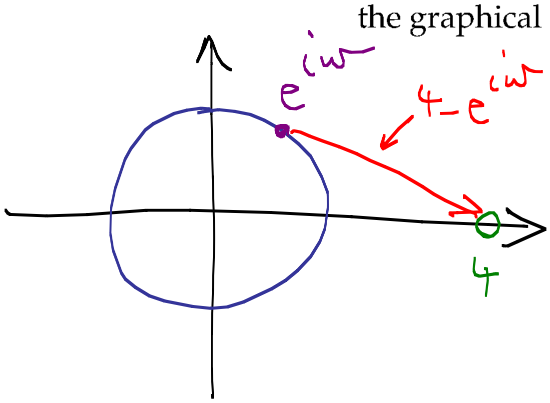
(ii) Determine an expression, as well as a well-labeled stem plot, for the composite system's impulse response $h(n)$.

Looking at $H(\omega) = \sum_n h(n) e^{-i\omega n}$ we note that only two terms are nonzero: $n = 0$, $n = +1$. $H(\omega) = h(0) + h(1)e^{-i\omega} = -1 + 4e^{-i\omega}$



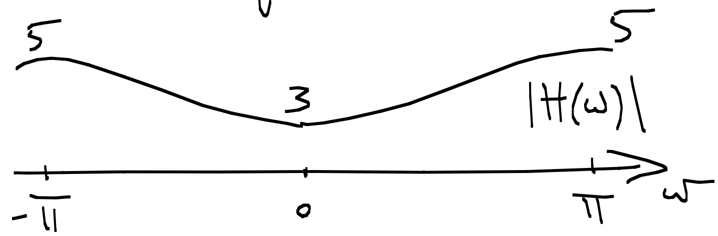
$$h(n) = -\delta(n) + 4\delta(n-1)$$

(iii) Provide a well-labeled plot of the composite system's magnitude response $|H(\omega)|$. Explain how you get your plot. (Again, we recommend the graphical method taught in lecture).



$$H(\omega) = 4e^{-i\omega} - 1 = \frac{4 - e^{i\omega}}{e^{i\omega}} \Rightarrow |H(\omega)| = |4 - e^{i\omega}|$$

$|H(\omega)|$ has exactly the same shape as $|F(\omega)|$, but scaled by a factor of 4.



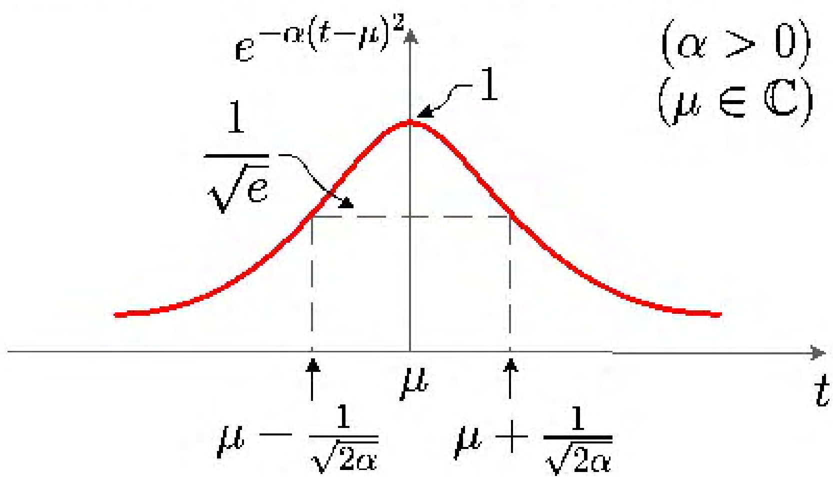
(iv) Explain a benefit of placing the original filter F in the feedback configuration H. Do this in *one* of two ways. Either compare the magnitude response plots $|F(\omega)|$ and $|H(\omega)|$, or determine the response of the composite system H to the input signal $1 + (-1)^n$ and compare it with the response of F to the same signal.

- Feedback gives us a factor-4 amplification in gain: $|H(\omega)| = 4\,|F(\omega)|$
- $1 + (-1)^n \rightarrow$ [H] $\rightarrow 3 + 5(-1)^n = 4\left[\frac{3}{4} + \frac{5}{4}(-1)^n\right]$

  (3 ← $H(0)$, 5 ← $H(\pi)$; $\frac{3}{4} + \frac{5}{4}(-1)^n$ = response of F to the same input.)

MT2.2 (50 Points) In this problem we explore some of the properties of a class of functions called *Gaussians* (the famous "bell-shaped" curves). Gaussian functions play an important role not only in probability and statistics, but also in signal processing and communication theory.

The figure below shows a Gaussian function of the form $g(t) = e^{-\alpha(t-\mu)^2}$.



The parameter μ is the center of symmetry of the function; in probability theory, it is called the *mean*. Though the figure is drawn for a real-valued μ, this parameter is allowed to be complex.

The parameter α describes the rate of decay of the Gaussian function as $|t| \to \infty$; throughout this problem, assume $\alpha > 0$. In probability theory, α is related to the *standard deviation*.

The area under the Gaussian function is given by the following integral:

$$\int_{-\infty}^{+\infty} e^{-\alpha(t-\mu)^2} dt = \sqrt{\frac{\pi}{\alpha}}.$$

Note that the area does *not* depend on μ. For example, the zero-mean Gaussian also has the same area.

$$\int_{-\infty}^{+\infty} e^{-\alpha t^2} dt = \sqrt{\frac{\pi}{\alpha}}.$$

(a) Consider a continuous-time LTI system F whose impulse response f is the Gaussian function

$$f(t) = e^{-\alpha t^2}.$$

Note that $\mu = 0$ for this function f. We call F a *Gaussian filter*.

(i) Show that the Gaussian filter's frequency response is also a Gaussian function, given by

$$F(\omega) = \sqrt{\frac{\pi}{\alpha}} \exp\left(-\frac{\omega^2}{4\alpha}\right).$$

You will find the method of *completing the square* helpful, whereby you can write

$$t^2 - 2bt = (t-b)^2 - b^2,$$

for some appropriately chosen b.

In this case, plug in the expression for $f(t)$ in

$$F(\omega) = \int_{-\infty}^{+\infty} f(t)\, e^{-i\omega t}\, dt\,,$$

complete the square in the exponent, and use an area formula given on the previous page.

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = \int_{-\infty}^{\infty} e^{-\alpha t^2 - i\omega t} dt = \int_{-\infty}^{\infty} e^{-\alpha\left(t^2 - \underbrace{\frac{i\omega}{\alpha}}_{2b} t\right)} dt \qquad b = \frac{i\omega}{2\alpha}$$

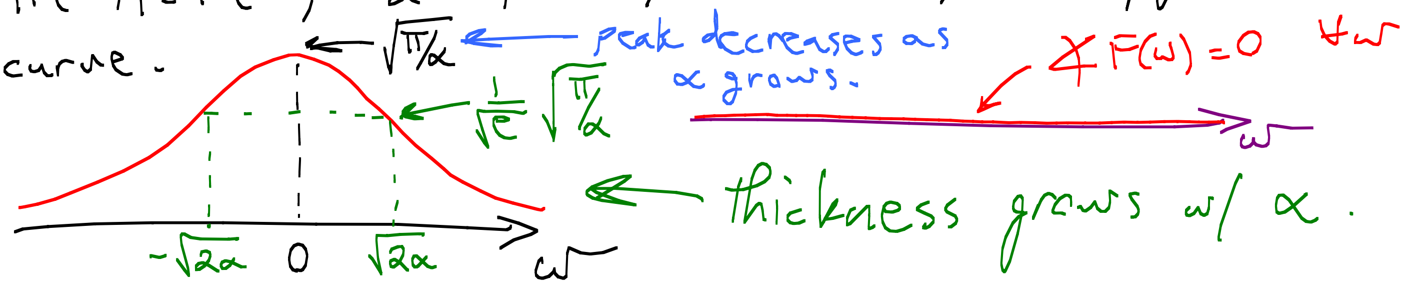
$$= \int_{-\infty}^{\infty} e^{-\alpha\left[\left(t - \frac{i\omega}{2\alpha}\right)^2 - \frac{i^2\omega^2}{4\alpha^2}\right]} dt = \underbrace{\left[\int_{-\infty}^{\infty} e^{-\alpha\left(t - \frac{i\omega}{2\alpha}\right)^2} dt\right]}_{\sqrt{\pi/\alpha}} e^{-\alpha\frac{\omega^2}{4\alpha^2}}$$

$$F(\omega) = \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\omega^2}{4\alpha}}$$

Also a Gaussian

(ii) Provide well-labeled plots of the Gaussian filter's magnitude response $|F(\omega)|$ and phase response $\angle F(\omega)$.

We note that $F(\omega) \in \mathbb{R}$ & $F(\omega) > 0 \;\; \forall \omega \implies$
$|F(\omega)| = F(\omega)$ & $\angle F(\omega) = 0$. $F(\omega)$ is simply a bell-shaped curve.



(iii) Explain how the time-frequency uncertainty plays out for this filter. In particular, how do the impulse response and frequency response change shape when α is increased. How do they change shape when α is decreased? Remember, $\alpha > 0$.

α increases $\longrightarrow f(t) = e^{-\alpha t^2}$ decays faster (bell gets thinner)
$\longrightarrow F(\omega) = \sqrt{\frac{\pi}{\alpha}}\, e^{-\omega^2/4\alpha}$ decays more slowly and peak at $\omega = 0$ gets shorter.

The opposite happens as α decreases.

Contraction in time $\longleftrightarrow$ Dilation in Frequency
Dilation in time $\longleftrightarrow$ Contraction in Frequency

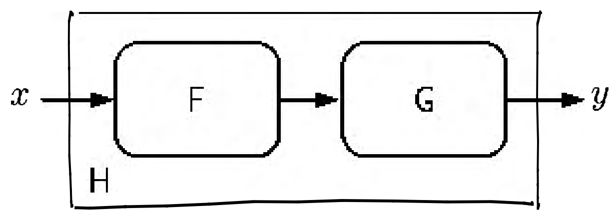
(iv) Determine the response of the filter if the input signal is constant, $x(t) = 1$ for all t.

$y(t) = F(0) \cdot 1 = \sqrt{\frac{\pi}{\alpha}} \implies y(t) = \sqrt{\pi/\alpha} \;\; \forall t$

(iv) True or False? The Gaussian filter F is causal. Explain.

False. An LTI filter F is causal iff its impulse response $f(t) = 0 \;\; \forall t < 0$. Clearly, this is not the case for the Gaussian filter.

(b) Suppose the Gaussian filter F is placed in a cascade configuration with another Gaussian filter G, as shown in the figure below.



The filter G has impulse response $g(t) = e^{-\beta t^2}$.

(i) Show that the frequency response $H(\omega)$ of the cascade interconnection is also a Gaussian, namely,

$$H(\omega) = \frac{\pi}{\sqrt{\alpha\beta}} \exp\left[-\frac{\omega^2}{4(\alpha \parallel \beta)}\right],$$

where we've borrowed from circuit theory vernacular and let

$$\alpha \parallel \beta = \frac{\alpha\beta}{\alpha+\beta} \quad \text{or equivalently} \quad \frac{1}{\alpha \parallel \beta} = \frac{1}{\alpha} + \frac{1}{\beta}.$$

$$H(\omega) = F(\omega)\,G(\omega) = \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\omega^2}{4\alpha}} \sqrt{\frac{\pi}{\beta}}\, e^{-\frac{\omega^2}{4\beta}}$$

$$= \frac{\pi}{\sqrt{\alpha\beta}}\, e^{-\frac{\omega^2}{4}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)} = \frac{\pi}{\sqrt{\alpha\beta}}\, e^{-\frac{\omega^2}{4}\left(\frac{\alpha+\beta}{\alpha\beta}\right)} = \frac{\pi}{\sqrt{\alpha\beta}}\, e^{-\frac{\omega^2}{4(\alpha\parallel\beta)}}$$

$$\Rightarrow H(\omega) = \frac{\pi}{\sqrt{\alpha\beta}}\, e^{-\frac{\omega^2}{4(\alpha\parallel\beta)}}$$

This is Gaussian.

(ii) Without much work, determine a reasonably simple expression for $h(t)$, the impulse response of the cascade system H.

Recall $f(t) = e^{-\alpha t^2} \longleftrightarrow F(\omega) = \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\omega^2}{4\alpha}}$

Here we have $H(\omega) = \frac{\pi}{\sqrt{\alpha\beta}} \exp\left[\frac{-\omega^2}{4(\alpha\|\beta)}\right] = \frac{\sqrt{\pi(\alpha\|\beta)}}{\sqrt{\alpha\beta}} \sqrt{\frac{\pi}{\alpha\|\beta}} \exp\left[\frac{-\omega^2}{4(\alpha\|\beta)}\right]$

$$H(\omega) = \sqrt{\frac{\pi}{\alpha+\beta}} \underbrace{\sqrt{\frac{\pi}{\alpha\|\beta}} \exp\left[-\frac{\omega^2}{4(\alpha\|\beta)}\right]}_{\text{maps to } \exp[-(\alpha\|\beta)t^2]} \Rightarrow h(t) = \sqrt{\frac{\pi}{\alpha+\beta}} \exp\left[-(\alpha\|\beta)t^2\right]$$

(c) True or False? Explain each of your answers.

(i) The set of all Gaussian functions is closed under multiplication. That is, multiplying two Gaussian functions produces another Gaussian function.

True. Take two zero-mean Gaussians. $f(t) = e^{-\alpha t^2}$, $g(t) = e^{-\beta t^2}$

$f(t)g(t) = e^{-(\alpha+\beta)t^2}$ which is another zero-mean Gaussian.

If they have different means, this would still be true. Without loss of generality assume one has zero mean, and the other same non-zero mean μ.

$f(t)g(t) = e^{-\alpha t^2} e^{-\beta(t-\mu)^2} = \cdots = K\exp\left[-\gamma(t-\lambda)^2\right]$, $K = \exp\left[-(\alpha\|\beta)\mu^2\right]$, $\gamma = \alpha+\beta$, $\lambda = \frac{\beta\mu}{\alpha+\beta}$

(ii) The set of all Gaussian function is closed under convolution.

True. Convolution in time is tantamount to multiplication in frequency. Since Gaussians are closed under multiplication, and since a Gaussian in frequency corresponds to a Gaussian in time, then the set of Gaussians must be closed under convolution.

$\mathcal{G}$ = set of Gaussians.

$$\left.\begin{array}{l} f \in \mathcal{G} \Rightarrow F \in \mathcal{G} \\ g \in \mathcal{G} \Rightarrow G \in \mathcal{G} \end{array}\right\} \Rightarrow FG \in \mathcal{G} \Rightarrow f * g \in \mathcal{G}$$

LAST Name Gauss FIRST Name Ian

Lab Time ?

Problem	Points	Your Score
Name	10	10
1	55	55
2	50	50
Total	**115**	115