MATH 1B MIDTERM 2 (LEC 001)
PROFESSOR PAULIN

INSTRUCTIONS

- Do not turn over until instructed to do so.
- **Write your name and SID in the spaces provided on one side of every page of the exam.**
- This exam consists of 5 questions.
- You have 60 minutes to complete this exam.
- This exam will be electronically scanned. Do not add or remove any pages from the exam.
- There is an extra blank page for scratch work on the back of the exam. It can also be used as extra space to write formal solutions as long as everything is clearly labeled.
- Calculators are not permitted.
- Show as much working as possible. Even if you don't end up with the correct answer, you may still get partial credit. Answers without justification will be viewed with suspicion and will not receive credit.
- You will find a simple formula sheet on the back of this page.

Name: ______________________________

Student ID: ______________________________

GSI Name: ______________________________

Formulae

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e \qquad |R_N(x)| \leq \frac{M_N |x-a|^{N+1}}{(N+1)!}$$

Name: SID:

1. (30 points) Determine the convergence or divergence of the following infinite series:

(a)

$$\sum_{n=1}^{\infty}(-1)^n\frac{3}{2n-1}$$

Solution:

$$\lim_{n\to\infty}\frac{3}{2n-1}=0$$

and

$$\left\{\frac{3}{2n-1}\right\} \text{ decreasing}$$

$$\underset{\text{A.S.T.}}{\Rightarrow} \sum_{n=1}^{\infty}(-1)\frac{3}{2n-1} \ \underline{\text{conv}}$$

(b)

$$\sum_{n=1}^{\infty} n^3\tan(\frac{4}{n^4})$$

Solution:

L'Hospital

$$\lim_{x\to 0}\frac{\tan(x)}{x}=\lim_{x\to 0}\frac{\sec^2(x)}{1}=1>0$$

$\div$

$$\lim_{n\to\infty}\frac{n^3\tan(\frac{4}{n^4})}{(\frac{4}{n})}=\lim_{n\to\infty}\frac{\tan(\frac{4}{n^4})}{(\frac{4}{n^4})}=1>0$$

$$\sum_{n=1}^{\infty}\frac{4}{n} \ \underline{\text{div}} \ \underset{\text{C.C.T.}}{\Rightarrow} \sum_{n=1}^{\infty}n^3\tan(\frac{4}{n^4}) \ \underline{\text{div}}$$

$p=1$

2. (30 points) Determine the **radius of convergence** of the following power series.

$$\sum_{n=1}^{\infty} \frac{n^n(2x+1)^n}{\underbrace{1\cdot 5\cdot 9\cdots(4n-3)}_{a_n}}$$

Solution:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left| \frac{\left(\frac{(n+1)^{n+1}(2x+1)^{n+1}}{1\cdot 5\cdot 9\cdots(4n-3)(4n+1)}\right)}{\left(\frac{n^n(2x+1)^n}{1\cdot 5\cdot 9\cdots\cdots(4n-3)}\right)} \right|$$

$$= \frac{n+1}{4n+1}\cdot\left(\frac{n+1}{n}\right)^n |2x+1|$$

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{e}{4}|2x+1|$$

Ratio Test $\Rightarrow$ $\begin{cases} \underline{\text{conv}} & \text{if } \frac{e}{4}|2x+1| < 1 \\ \underline{\text{div}} & \text{if } \frac{e}{4}|2x+1| > 1 \end{cases}$

$$\frac{e}{4}|2x+1| < 1 \iff |2x+1| < \frac{4}{e} \iff 2\left|x+\frac{1}{2}\right| < \frac{4}{e}$$

$$\Rightarrow \left|x+\frac{1}{2}\right| < \frac{2}{e} \Rightarrow \text{R.O.C.} = \frac{2}{e}$$

3. (30 points) Using the **integral test**, and any other relevant tests, determine whether the following infinite series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{(n^2+1)^2}$$

Be sure to check that all appropriate conditions hold.

Solution:

$+/+$ on $[1,\infty)$

$$f(x)=\frac{x}{(x^2+1)^2} \Rightarrow f'(x)=\frac{1\cdot(x^2+1)^2-4x^2(x^2+1)}{(x^2+1)^4}$$

Denominator $\neq 0$ on $[1,\infty)$

$$=\frac{-3x^4-2x^2+1}{(x^2+1)^2}$$

$-/+$ ← negative on $[1,\infty)$

$\Rightarrow$ $f(x)$ positive, continuous, decreasing on $[1,\infty)$

$$u=x^2+1 \Rightarrow \frac{du}{dx}=2x \Rightarrow dx=\frac{du}{2x}$$

$$\Rightarrow \int f(x)\,dx=\int\frac{1}{2u^2}\,du=\frac{-1}{2u}+C=\frac{-1}{2(x^2+1)}+C$$

$$\Rightarrow \int_1^{\infty}f(x)\,dx=\lim_{t\to\infty}\left.\frac{-1}{2(x^2+1)}\right|_1^t=\lim_{t\to\infty}\frac{1}{4}-\frac{1}{2(t^2+1)}=\frac{1}{4}$$

$$\int_1^{\infty}f(x)\,dx \ \underline{\text{conv}} \ \overset{\Rightarrow}{\text{I.T.}} \ \sum_{n=1}^{\infty}\left|(-1)^n\frac{n}{(n^2+1)^2}\right| \ \underline{\text{conv}}$$

$$\Rightarrow \sum_{n=1}^{\infty}(-1)^n\frac{n}{(n^2+1)^2} \text{ absolutely convergent}$$

Name: SID:

4. (30 points) Find a power series (centered at 1) that represents the following function on an open interval containing 1.

$$f(x) = \frac{xe^x}{e}$$

Carefully justify your answer and be sure to include a general term.

What is the value of $f^{(2025)}(1)$

Solution:

$(-\infty, \infty)$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

Replace x with $x-1$

$(-\infty, \infty)$

$$\frac{e^x}{e} = e^{x-1} = 1 + \frac{(x-1)}{1!} + \frac{(x-1)^2}{2!} + \cdots + \frac{(x-1)^n}{n!} + \cdots$$

Multiply by $x-1$

$(-\infty, \infty)$

$$(x-1)\frac{e^x}{e} = (x-1) + \frac{(x-1)^2}{1!} + \frac{(x-1)^3}{2!} + \cdots + \frac{(x-1)^n}{(n-1)!} + \cdots$$

Sum

$(-\infty, \infty)$

$$\frac{xe^x}{e} = 1 + \sum_{n=1}^{\infty}\left[\frac{1}{(n-1)!} + \frac{1}{n!}\right](x-1)^n$$

$$\frac{f^{(2025)}(1)}{2025!} = \frac{1}{2024!} + \frac{1}{2025!}$$

$$\Rightarrow f^{(2025)}(1) = 2025!\left[\frac{1}{2024!} + \frac{1}{2025!}\right] = 2026$$

Name: SID:

5. (30 points) Show that the polynomial function $-2x - 2x^2$ approximates the function $f(x) = \ln(1-2x)$ to within $\frac{1}{3}$ for all x in $[-1/4, 1/4]$. Carefully justify your answer.

Solution:

$(-1,1)$

$$\ln(1+x) = x - \frac{x^2}{2} \cdots$$

$(-\frac{1}{2}, \frac{1}{2})$

$$\Rightarrow \ln(1-2x) = -2x - \frac{4x^2}{2} \cdots$$ Maclaurin series

$$\Rightarrow T_2(x) = -2x - 2x^2$$ $n=2$

$$f'(x) = \frac{-2}{1-2x} \Rightarrow f''(x) = \frac{-2^2}{(1-2x)^2} \Rightarrow f'''(x) = \frac{-2^4}{(1-2x)^3}$$ + decreasing on $[-\frac{1}{4}, \frac{1}{4}]$

$$\Rightarrow |f'''(x)| \leq \frac{2^4}{(1-2\cdot\frac{1}{4})^3} = \frac{2^4}{(\frac{1}{2})^3} = 2^7$$ on $[-\frac{1}{4}, \frac{1}{4}]$, $a=0$, $d=\frac{1}{4}$

$$\Rightarrow |\ln(1-2x) - (-2x-2x^2)| \leq \frac{2^7(\frac{1}{4})^3}{3!} = \frac{2}{3!} = \frac{1}{3}$$

T.I.

on $[-\frac{1}{4}, \frac{1}{4}]$

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END OF EXAM