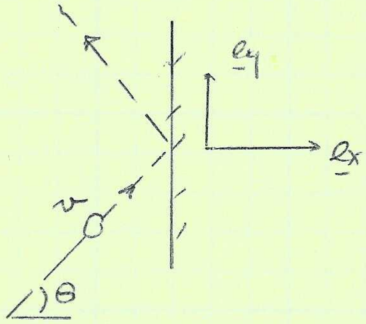
Midterm 2 Solutions

1.



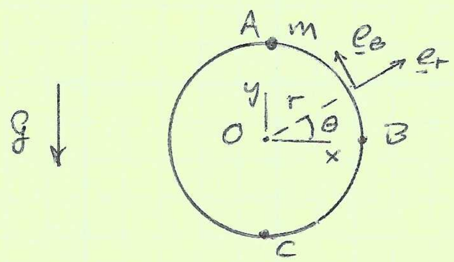
$\underline{t} = \pm \underline{e}_y,$
$\underline{n} = \pm \underline{e}_x.$

Just before impact, $\underline{v} = v(\cos\theta\, \underline{e}_x + \sin\theta\, \underline{e}_y)$.

Just after impact (from eq. (*) on p. 97 of Notes),

$$\underline{v}' = (\underline{t}\cdot\underline{v})\underline{t} - e(\underline{n}\cdot\underline{v})\underline{n}$$
$$= (\underline{e}_y\cdot\underline{v})\underline{e}_y - e(\underline{e}_x\cdot\underline{v})\underline{e}_x$$
$$\Rightarrow \quad \underline{\underline{v}' = v\sin\theta\, \underline{e}_y - e\, v\cos\theta\, \underline{e}_x}.$$

2.



smooth ring - no friction.
$\Rightarrow$ conservative problem.

$E = K + U$, where $K = \frac{1}{2}mv^2$ and $U = mgy$ (+const.)

$E_A = E_B \Rightarrow \frac{1}{2}mv_B^2 = mgr \Rightarrow \underline{v_B = \sqrt{2gr}}$

$E_A = E_C \Rightarrow \frac{1}{2}mv_C^2 - mgr = mgr \Rightarrow \underline{v_C = \sqrt{4gr} = 2\sqrt{gr}}.$

(check: $E_B = E_C \Rightarrow \frac{1}{2}mv_C^2 - mgr = \frac{1}{2}mv_B^2 \Rightarrow v_C^2 = v_B^2 + 2gr$
$= 4gr$ ✓).

$$\underline{H}_0 = \underline{r} \times m\underline{v} = r\underline{e}_r \times m v \underline{e}_t$$
$$= rmv\, \underline{e}_r \times (-\underline{e}_\theta) = -rmv\,\underline{k}.$$

$\Rightarrow \quad |\underline{H}_0| = rmv.$

(a) $\underline{|\underline{H}_0|_B} = rmv_B = \underline{rm\sqrt{2gr}}.$

(b) $\underline{|\underline{H}_0|_C} = rmv_C = \underline{rm\sqrt{4gr}}.$

2, continued:

(c) $\dot{\underline{H}}_o = (\underline{r} \times m\underline{v})^{\cdot} = \underbrace{\underline{v} \times m\underline{v}}_{\underline{0}} + \underline{r} \times m\dot{\underline{v}}$

$= \underline{r} \times m\underline{a} = \underline{r} \times \underline{F}$,

where $\underline{F} = N\underline{e}_r - mg\underline{e}_y$ (N = normal force by track on mass - no friction).

Thus,

$$\dot{\underline{H}}_o = r\underline{e}_r \times (N\underline{e}_r - mg\underline{e}_y)$$

$$= -mgr\, \underline{e}_r \times \underline{e}_y$$

$$= -mgr(\cos\theta\, \underline{e}_x + \sin\theta\, \underline{e}_y) \times \underline{e}_y$$

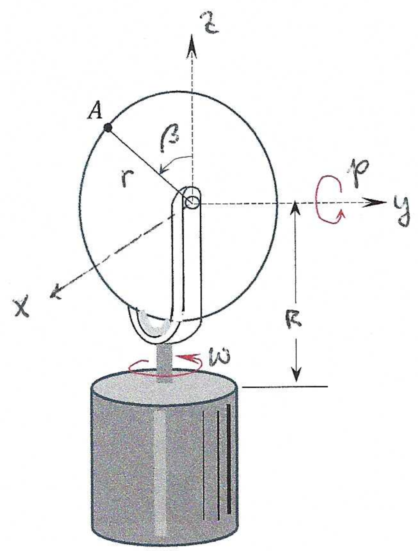
$$= -mgr\cos\theta\, \underline{k}.$$

at point B, $\theta = 0 \Rightarrow \dot{\underline{H}}_o = -mgr\underline{k}$

at point C, $\theta = -\frac{\pi}{2} \Rightarrow \dot{\underline{H}}_o = \underline{0}$.

3.

A circular disk of radius r is mounted in a bearing in a yoke and spins with constant angular velocity $\dot{\beta} = p$. The yoke also revolves with constant angular velocity ω counterclockwise as shown. Find $\boldsymbol{a}_A$ in terms of the angle β.



Solution

The inertial reference system[1] is fixed to the ground. The rotating system axes are labeled in the figure as xyz. They are attached to the yoke at the center of the spinning disk and rotate at angular velocity $\omega\hat{\boldsymbol{k}}$. Note that the basis vectors associated with the xyz system are denoted as $(\hat{\imath}, \hat{\jmath}, \hat{\boldsymbol{k}})$.

Point A on the disk moves so that it has a velocity relative to the rotating system.

$$\boldsymbol{r}_{AB} = r\sin\beta\,\hat{\imath} + r\cos\beta\,\hat{\boldsymbol{k}}$$

$$\boldsymbol{v}_{rel} = p\hat{\jmath} \times \boldsymbol{r}_{AB} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ 0 & p & 0 \\ r\sin\beta & 0 & r\cos\beta \end{vmatrix} = rp\cos\beta\,\hat{\imath} - rp\sin\beta\,\hat{\boldsymbol{k}}$$

$$\boldsymbol{a}_A = \overbrace{\dot{\boldsymbol{\omega}}}^{=0} \times \boldsymbol{r}_{AB} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \boldsymbol{r}_{AB} + 2\boldsymbol{\omega} \times \boldsymbol{v}_{rel} + \boldsymbol{a}_{rel}$$

$$\boldsymbol{\omega} \times \boldsymbol{r}_{AB} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ 0 & 0 & \omega \\ r\sin\beta & 0 & r\cos\beta \end{vmatrix} = \omega r\sin\beta\,\hat{\jmath}$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_{AB}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ 0 & 0 & \omega \\ 0 & \omega r\sin\beta & 0 \end{vmatrix} = -\omega^2 r\sin\beta\,\hat{\imath}$$

$$2\boldsymbol{\omega} \times \boldsymbol{v}_{rel} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ 0 & 0 & 2\omega \\ rp\cos\beta & 0 & -rp\sin\beta \end{vmatrix} = 2\omega rp\cos\beta\,\hat{\jmath}$$

$$\boldsymbol{a}_{rel} = p\hat{\jmath} \times (p\hat{\jmath} \times \boldsymbol{r}_{AB}) = p\hat{\jmath} \times \left[p\hat{\jmath} \times (r\sin\beta\,\hat{\imath} + r\cos\beta\,\hat{\boldsymbol{k}})\right] = p\hat{\jmath} \times (pr\cos\beta\,\hat{\imath} - pr\sin\beta\,\hat{\boldsymbol{k}})$$

$$= -p^2 r\cos\beta\,\hat{\boldsymbol{k}} - p^2 r\sin\beta\,\hat{\imath}$$

Note that $\boldsymbol{a}_{rel}$ is the acceleration of point A relative to the axes xyz that rotate at constant angular velocity $p\hat{\jmath}$.

$$\boldsymbol{a}_A = -\omega^2 r\sin\beta\,\hat{\imath} + 2\omega rp\cos\beta\,\hat{\jmath} - p^2 r\cos\beta\,\hat{\boldsymbol{k}} - p^2 r\sin\beta\,\hat{\imath}$$

$$\boldsymbol{a}_A = -(p^2 + \omega^2) r\sin\beta\,\hat{\imath} + 2\omega rp\cos\beta\,\hat{\jmath} - p^2 r\cos\beta\,\hat{\boldsymbol{k}}$$

[1] An inertia reference frame is one that is not accelerating.