INDENG 164: Introduction to optimization modeling
Spring 2025 Midterm
Instructor: Ying Cui

- This exam is closed notes and closed book. You **may not** use graphing calculators, laptops, cellphones, tablets, or any other electronic devices. You are allowed to bring one page formula sheet.
- Submit your hand-written notes along with the completed exam sheet.
- The exam contains 4 questions on pages 2-10, and pages 11-12 are blank for extra work if you need it. Each question is worth 10 points. You have 50 minutes.
- Please do not unstaple/tear the exam sheets apart. If you really must, then write your name on each separated sheet so that it is not lost.
- Make sure you answer the questions you find easy before spending a lot of time on the difficult ones.
- Read and sign the pledge **before** beginning the exam.

Academic integrity is expected of all students of University of California, Berkeley at all times. Understanding this, I declare that I shall not give, use or receive unauthorized aid in this examination.

Name: ______________________________ SID: ______________________________

Signature: ____

Q1 (10 pts). An airline needs to schedule pilots for three different flight routes. Each route and each pilot must adhere to certain regulations.

Route & Flight Information

- Each route may be scheduled for multiple flights every week.
- Each flight requires exactly one pilot.
- The duration and profit per flight for each route are given in the table below:

Route	Route Name	Flight Time (hours)	Profit per flight ($\$ \times 10^4$)
1	Domestic (Short-haul)	2	5
2	Transatlantic	8	20
3	Intercontinental	12	50

Constraints:

- The airline has 50 available pilots, with ID numbers $i = 1, \ldots, 50$.
- Each pilot can work a maximum of 40 hours per week.
- Each pilot must fly at least one domestic flight per week for route qualification.

Objective:

- Maximize the total profit of the airline in a week.

Let $x_{ij} \geq 0$ represent the number of flights that pilot i undertake for route j, where $i = 1, \cdots, 50$ and $j = 1, 2, 3$. Formulate a linear optimization model to find x_{ij} that maximizes the profit while ensuring all regulations are met. You can ignore the integrality constraints (i.e., we allow "half" flights if necessary).

Solutions:

$$
\begin{aligned}
\max_{x_{ij}} \quad & \sum_{i=1}^{50} (5x_{i1} + 20x_{i2} + 50x_{i3}) \\
\text{s.t.} \quad & 2x_{i1} + 8x_{i2} + 12x_{i3} \leq 40, \ \forall i = 1, \ldots, 50 && \text{(hour allowance)} \\
& x_{i1} \geq 1, \ \forall i = 1, \cdots, 50 && \text{(route qualification)} \\
& x_{ij} \geq 0, \ \forall i = 1, \cdots, 50, \ j = 1, 2, 3.
\end{aligned}
$$

Q2 (10 pts). A smart home system is designed to optimize energy usage by scheduling household appliances efficiently. Consider a house that has four appliances that consume different amounts of electricity and can only be operated during specific time windows. Information of the appliances is given below.

	Appliance	Energy Consumption (kWh)
1	Air Conditioner	3.5
2	Portable Heater	3.0
3	Washing Machine	1.5
4	Dryer	2.0

Constraints:

- If the washing machine is used, the dryer must also be used.
- It is chilly recently, so either Air Conditioner or Portable Heater must be used.
- To prevent electrical overload, no more than two high-power appliances (Air Conditioner, Portable Heater, Dryer) should be used on the same day.

The home runs under a tiered cost rate plan. The per-kWh cost is \$2 dollar for all energy usage, and the plan charges \$1 *additional* dollar for every unit energy consumed beyond 4kWh. For example, if in a day the home consumes 5kWh, the cost is $\$2 \times 5 + \$1 \times (5 - 4)$.

Formulate a mixed integer linear optimization model to decide whether each appliance should be turned on or off in a given day. The goal is to minimize total energy cost while satisfying all constraints.

Solutions: Let $x_i \in \{0, 1\}$ represent whether to turn on or off appliance i in a given day. Let $y \geq 0$ represent the units (kWh) of additional energy consumed beyond 4 kWh.

$$
\begin{aligned}
\min_{x_1, \cdots, x_4, y} \quad & 2(3.5x_1 + 3x_2 + 1.5x_3 + 2x_4) + y \\
\text{s.t.} \quad & x_3 \leq x_4 \\
& x_1 + x_2 \geq 1 \qquad \text{(fine to write } x_1 + x_2 = 1\text{)} \\
& x_1 + x_2 + x_4 \leq 2 \\
& y \geq (3.5x_1 + 3x_2 + 1.5x_3 + 2x_4) - 4, \ y \geq 0 \\
& x_i \in \{0, 1\}, \ \forall i = 1, 2, 3, 4
\end{aligned}
$$

Q3 (5+5 pts). UC Berkeley is installing public laptop charging stations across campus. The goal is to supply power from central power plants to different campus buildings while minimizing electricity costs. However, due to infrastructure limitations, some power must be routed through intermediate relay stations before reaching the buildings.

Network Structure

- Power Stations (P1, P2) generate electricity.
- Relay Stations (R1, R2) help distribute electricity.
- Campus Buildings (B1, B2, B3) are the final destinations for power.

Power Flow Network

Route	Capacity (kWh)	Cost per kWh ($)
Power Stations → Relay Stations		
P1 → R1	100	0.08
P1 → R2	80	0.10
P2 → R2	120	0.07
Relay Stations → Buildings		
R1 → B1	40	0.05
R1 → B3	70	0.06
R2 → B2	150	0.04
Power Stations → Buildings (Expensive Option)		
P1 → B1	50	0.12
P2 → B3	40	0.11

Supply/Demand Data

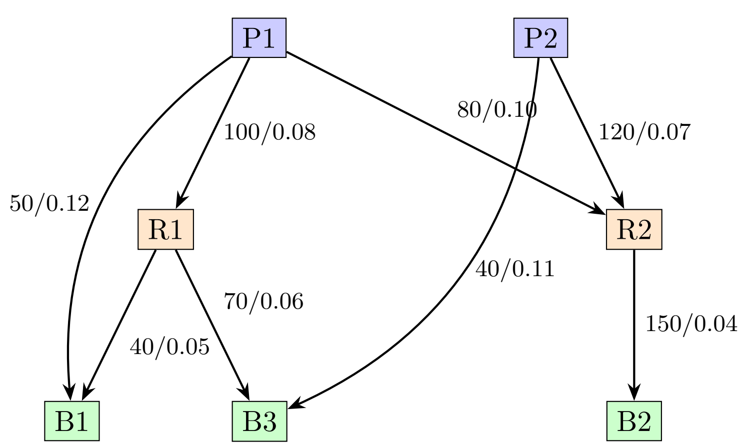
Supply at Power Stations:

- P1: 160 kWh
- P2: 160 kWh

Demand at Buildings:

- B1: 90 kWh
- B2: 130 kWh
- B3: 100 kWh

(a) Draw a network diagram. Illustrate the nodes (power stations, relay stations, and buildings) and edges (power transmission routes) of the problem as a directed graph. Label each edge with its capacity and cost per kWh. For example, you can format the label: $\text{P1} \xrightarrow{100/0.08} \text{R1}$.



(b) Formulate a minimum-cost flow optimization model to minimize total power distribution cost while ensuring all campus buildings receive their required power (no shortages allowed). Recall that a minimum-cost flow model can be written in the general form of:

$$
\begin{aligned}
\min_{x_{i,j}} \quad & \sum_{(i,j)} c_{i,j} x_{i,j} \\
\text{s.t.} \quad & Ax = b \\
& 0 \le x_{i,j} \le u_{i,j} \quad \text{for all } (i,j) \text{ pairs}
\end{aligned}
$$

where $c_{i,j}$ and $u_{i,j} \in \mathbb{R}^8$ represents the cost and capacity associated with each route respectively, and $x := (x_{i,j}) \in \mathbb{R}^8$ are the decision variables representing the amount of power delivered from node i to node j. For example, x_{P_1,R_1} represents the flow from node P_1 to node R_1.

Please fill in the matrix A and vector b below to correctly represent the flow balance constraint $Ax = b$. You may leave an entry empty if it is filled with zero. Each row of A and b represents the flow balance at one node. For example, row 1 represents the flow balance at node P_1.

$$
\overbrace{
\begin{array}{c}
\begin{array}{cccccccc}
x_{P_1,R_1} & x_{P_1,R_2} & x_{P_2,R_2} & x_{R_1,B_1} & x_{R_1,B_3} & x_{R_2,B_2} & x_{P_1,B_1} & x_{P_2,B_3}
\end{array} \\
\begin{array}{c|cccccccc|}
P_1 & 1 & 1 & & & & & 1 & \\
P_2 & & & 1 & & & & & 1 \\
R_1 & -1 & & & 1 & 1 & & & \\
R_2 & & -1 & -1 & & & 1 & & \\
B_1 & & & & -1 & & & -1 & \\
B_2 & & & & & & -1 & & \\
B_3 & & & & & -1 & & & -1
\end{array}
\end{array}
}^{\text{node-arc incidence matrix } A}
\begin{bmatrix}
x_{P_1,R_1} \\ x_{P_1,R_2} \\ x_{P_2,R_2} \\ x_{R_1,B_1} \\ x_{R_1,B_3} \\ x_{R_2,B_2} \\ x_{P_1,B_1} \\ x_{P_2,B_3}
\end{bmatrix}
=
\begin{array}{c}
b_i \\
\begin{array}{c|c|}
P_1 & 160 \\
P_2 & 160 \\
R_1 & \\
R_2 & \\
B_1 & -90 \\
B_2 & -130 \\
B_3 & -100
\end{array}
\end{array}
$$

Q4 (10 pts). A college student wants to invest in three assets: stocks, bonds, and a tech startup fund. The goal is to allocate money to maximize expected return, subject to budget and risk constraints. The expected return of per dollar investment is given in the following table.

Asset	Investment dollar	Expected Return per dollar investment
Stocks	x_1	8
Bonds	x_2	5
Tech Fund	x_3	12

Constraints:

- The total investment cannot exceed $\$10,000$.
- The expected return ratio between Stocks and Bonds has to be at least 2.
- The total expected return of Tech Fund has to be at least $\$50,000$.

Objective: Minimize the total portfolio risk, modeled as

$$x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + x_1x_3 + 10x_2x_3.$$

Notice that this is not a convex function of x_1, x_2, x_3.

Formulate a convex optimization model to determine the optimal investment amounts in stocks, bonds and tech fund that maximize the expected return. To get a convex formulation, you may consider to transform the decision variables via $z_i = \log(x_i)$, and use the fact that the exponential function e^{z_i} and the log-sum-exp function $\log\left(\sum_{i=1}^{n} e^{z_i}\right)$ are both convex functions of z_i's.

Solutions:

$$\begin{array}{ll} \min_x & x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 + x_1x_3 + 10x_2x_3 \\ \text{s.t.} & x_1 + x_2 + x_3 \le 10000 \\ & \frac{8x_1}{5x_2} \ge 2 \\ & 12x_3 \ge 50,000 \\ & x_1, x_2, x_3 \ge 0. \end{array}$$

Define z_i such that $x_i = \exp(z_i)$.

$$\begin{array}{ll} \min\limits_z & e^{2z_1} + 2e^{2z_2} + e^{2z_3} + e^{z_1+z_2} + e^{z_1+z_3} + e^{10z_2+10z_3} \\ \text{s.t.} & e^{z_1} + e^{z_2} + e^{z_3} \le 10,000 \\ & \log(e^{z_1-z_2}) \ge \log(10/8) \qquad \text{(equiv. to } z_1 - z_2 \ge \log(10/8)) \\ & z_3 \ge \log(5000/12) \end{array}$$

(Additional space for rough work)

(Additional space for rough work)